Instituto de Física Teórica UAM-CSIC

Universidad Autónoma de Madrid

Planar Radiation Zeros and Scattering Equations in Field Theory Amplitudes

PhD thesis dissertation
by
Diego Medrano Jiménez

Thesis presented to the "Universidad Autónoma de Madrid" in fulfillment of the requirements for the degree of *Doctor of Philosophy* in Theoretical Physics. Supervised by Prof. A. Sabio Vera and Prof. M. Á. Vázquez-Mozo.

Madrid, September 2019

To Alba . . .

Acknowledgments

Firstly, I would like to express my sincere gratitude to Agustín Sabio Vera for the support, patience and especially for the continuous motivation throughout the whole process.

I would like to thank as well Tristan McLoughlin for the great hospitality during my research stay at the "Trinity College of Dublin", and Riccardo Gonzo for the opportunity to join the team.

Last but not least, friend and partner of adventures, I would like to thank David Gordo for hundreds of anecdotes that we have collected during many summer schools and workshops all around Europe.

Abstract / Resumen

We have presented for the first time a detailed description of *planar radiation zeros* as a novel mathematical structure giving rise to new insights on the internal behavior of a theory, such as the biadjoint scalar theory, the Yang-Mills theory or the Einstein-Hilbert gravity. The concept “radiation zero” makes reference to all the configurations in phase space for which the full scattering amplitude of a given process vanishes. In our case, we have studied “planar zeros”, meaning that our characterization applies to those processes where all particle momenta lie in the same spatial plane. Although being a rather naive concept, the obtained results are far from incidental. On one side, we have found that the conditions of emergence of gauge planar zeros in the *maximally helicity violating* sector live inside the projective space spanned by the stereographic coordinates labelling the direction of flight of the outgoing momenta. The existence of such a projective characterization implies that planar zeros are always realized inside the soft limit of any of the emitted particles, which might be of relevance for the infrared structure or the asymptotic symmetries of the theory. On a different side, we have found that gravitational amplitudes always vanish inside this planar limit for non-helicity conserving configurations without imposing any further kinematic conditions. String α'-corrections of these behaviors have also been obtained. All the computations have been done in the context of the *color-kinematics duality*, used as a procedure to compute gravitational amplitudes from their gauge analogues; and the *Cachazo-He-Yuan formalism*, as a novel integral representation to write scattering amplitudes in contrast to the traditional Feynman diagram decomposition. In particular, the latter relies upon a rational map between the space of null D-dimensional momentum vectors and the moduli space of punctured Riemann spheres, given the name of *scattering equations*. Considered to be a challenging task, we have shown the advantages of using the *Sudakov parametrization* of particle momenta to simplify the computation of their exact solutions. In particular, we have shown that both punctures in the Riemann sphere and scattering amplitudes themselves adopt rather compact formulas when expressed in terms of Sudakov variables, suggesting the parametrization to be a natural candidate for an efficient description of scattering amplitudes inside the formalism.

* * *

Presentamos por primera vez una descripción detallada de los *ceros de radiación planares*, como una estructura matemática que da lugar a nuevos puntos de vista sobre el comportamiento interno de una teoría. El concepto de “cero de radiación” hace referencia a las configuraciones en el espacio de fases para las que la amplitud de dispersión completa de un proceso dado se anula. En nuestro caso, hemos estudiado “ceros planares”, lo que significa que hemos realizado nuestra caracterización para procesos en los cuáles todos los momentos de las partículas involucradas se encuentran confinados dentro de un mismo plano espacial. A pesar de ser un concepto muy sencillo, los resultados obtenidos están lejos de ser casuales. Por un lado, hemos encontrado que las condiciones de emergencia de ceros planares en teorías 'gauge' en el sector que viola helicidad maximalmente, se encuentran definidas dentro de un espacio proyectivo generado por las coordenadas estereográficas correspondientes a la dirección espacial de los momentos de las partículas finales. La existencia de esta caracterización en un espacio proyectivo implica que los ceros planares tienen lugar siempre en el límite en que una de las partículas emitidas tiene poca energía, lo que puede tener importancia para la estructura en el infrarrojo de la teoría o para el estudio de las simetrías asintóticas de la misma. Por otro lado, hemos encontrado que las amplitudes gravitatorias siempre se anulan en el límite planar para configuraciones que no conservan helicidad, sin la necesidad de imponer ninguna condición cinemática adicional. También se han obtenido correcciones en teoría de cuerdas de estos comportamientos. Todos los cálculos han sido realizados en el contexto de la *dualidad color-cinemática*, utilizada como un procedimiento para calcular amplitudes gravitatorias a partir de sus análogas en teorías 'gauge'; y el *formalismo de Cachazo-He-Yuan (CHY)*, como una representación novedosa con la que escribir amplitudes de dispersión en contraste con la descomposición más tradicional en diagramas de Feynman. Este formalismo está fuertemente basado en un mapa entre el espacio de momentos nulos D-dimensionales y el espacio modular de esferas de Riemann con punturas, conocido como *ecuaciones de dispersión (SE)*. A pesar de ser una tarea ardua, hemos mostrado por primera vez las ventajas de utilizar la *parametrización de Sudakov* en el espacio de momentos para simplificar el cálculo de sus soluciones exactas. En particular, hemos mostrado que tanto las punturas como las propias amplitudes de dispersión dan lugar a fórmulas compactas cuando se expresan en función de variables de Sudakov. Esto sugiere que la parametrización es una candidata natural para una descripción eficiente de las amplitudes de dispersión dentro del formalismo de CHY.

Contents

Author's declaration

I declare that this thesis is the result of my own original work and that it has not been submitted, in whole or in part, for any other degree or academic award. In cases where the work of others is presented, appropriate citations are used. The thesis is based on the following publications:

- [1] D. Medrano Jiménez, A. Sabio Vera and M. Á Vázquez-Mozo, *Planar Zeros in Gauge Theories and Gravity*, *JHEP* **09** (2016) 006 [1607.04605].
- [2]. D. Medrano Jiménez, A. Sabio Vera and M. Á. Vázquez-Mozo, *Projectivity of Planar Zeros in Field and String Theory Amplitudes*, *JHEP* **05** (2017) 011 [1703.07274].
- [3] G. Chachamis, D. Medrano Jimenez, A. Sabio Vera and M. A. Vazquez-Mozo, *Sudakov Representation of the Cachazo-He-Yuan Scattering Equations Formalism*, *JHEP* **01** (2018) 057 [1712.04288].

PREFACE

After their manifest relevance during the first decades of the XX century, Quantum Mechanics (QM) and Special Relativity (SR) were fused together giving rise to what is known as relativistic Quantum Field Theory (QFT). Provided with a rich mathematical structure, it rapidly became the main theoretical framework for the description of elementary particles and their interactions. The cornerstone of this construction, allowing for the direct connection of theoretical predictions and experiment, is the computation of scattering amplitudes.

Traditionally, Feynman diagrams have been the standard approach to compute them. Starting from a collection of rules derived from the lagrangian of the theory under study, amplitudes are written directly as a sum of terms corresponding to a graphical representation of the process. Locality and unitarity, being two important principles underlying particle interactions, are manifest in this description. However, there is an unavoidable drawback in the procedure: the presence of gauge redundancies associated to unphysical degrees of freedom. Although the full amplitude is free from these redundancies, each of the diagrams in the sum suffers from an excess of information. As a consequence, usual Feynman diagram techniques become too complicated as the number of external legs or loops increases and therefore any alternative strategies are always desirable.

The progress in the understanding and calculation of scattering amplitudes in recent years has been enormous (see [4–9] for a general review on the main topics of the field). The reason is bifold. On one side, in contrast to Feynman diagrams, efficient methods are always necessary from the phenomenological point of view in order to have more accurate predictions to test the validity of the Standard Model (SM) or to find any evidence of new physics. On a more formal side, it has been found that scattering amplitudes often manifest hidden symmetries besides those present in the lagrangian, becoming an interesting object of study *per se* to deepen the understanding of a theory. In both cases then, it is worth exploring their mathematical structure, treating scattering amplitudes simply as analytical functions and exploiting their properties for the sake of alternative descriptions and a more complete comprehension of the underlying symmetries in nature.

In most of these developments, Yang-Mills (YM) theory —and its maximally $\mathcal{N} = 4$ supersymmetric extension— serve as the perfect playground, being an idealized version of Quantum Chromodynamics (QCD), for testing new mathematical tools and representations. Hence, many of the results throughout the thesis will focus on pure gluon amplitudes. Any additional matter content or different scenarios will be refered explicitly in the text when needed. Let us now review a few examples of modern amplitude descriptions that are more advantageous than the standard QFT textbook methodology.

Taking into account the color structure of a general n-gluon process, it is possible to make use of the $SU(N_c)$ generator identities to decompose the full tree-level amplitude as a sum over single-traces of the color generators T^a in the following way

$$A_n^{\text{tree}}(\{a_i, p_i, h_i\}) = g^{n-2} \sum_{\sigma \in S_n/\mathbb{Z}_n} \text{tr}\,(T^{a_{\sigma_1}} T^{a_{\sigma_2}} \ldots T^{a_{\sigma_n}}) \\ \times A_n^{\text{tree}}(p_{\sigma_1}, h_{\sigma_1}; p_{\sigma_2}, h_{\sigma_2} \ldots ; p_{\sigma_n}, h_{\sigma_n}) \;, \qquad (0.0.1)$$

where the sum runs over non-cyclic permutations of the external legs. Notice that this representation of the amplitude has disentagled the color and kinematical degrees of freedom in a very specific way, where each of the orderings define now a color basis of *partial amplitudes* $A_n^{\text{tree}}[\sigma(1, \ldots, n)]$. By construction, these partial amplitudes contain information just about the kinematics of the process and are apparently simpler objects than the full amplitude, since they only contribute to a particular ordering of the gluons. They receive the name of *color-ordered amplitudes*. Some identities, inherited from the properties of the color traces, on which the decomposition in Eq. (0.0.1) is based upon are

- Cyclic property:

$$A_n^{\text{tree}}(1, 2, \ldots, n) = A_n^{\text{tree}}(2, \ldots, n, 1) \;. \qquad (0.0.2)$$

- Reflection property:

$$A_n^{\text{tree}}(1, 2, \ldots, n) = (-1)^n \, A_n^{\text{tree}}(n, \ldots, 2, 1) \;. \qquad (0.0.3)$$

- Subcyclic property:

$$\sum_{\sigma \in S_{n-1}} A_n^{\text{tree}}[1, \sigma(2, 3, \ldots, n)] = 0 \;. \qquad (0.0.4)$$

  This property is also know as $U(1)$ photon decoupling identity. It can be derived from Eq. (0.0.1) by substituting one of the gluons by a photon —i.e. setting one of the color generators $T^a \to \mathbb{I}$—, and imposing that such an amplitude vanishes.

It turns out then that the basis of partial amplitudes is overcomplete. For a long time it was thought that the identities mentioned above were enough to reduce the number of elements in the basis to $(n-2)!$, but this is only true for low multiplicities —i.e. $n = 4, 5, 6$—. The extra identity, derived in [10], allowing for this reduction for arbitrary multiplicities is

- Kleiss-Kuijf (KK) relations:

$$A_n^{\text{tree}}(1, \{\alpha\}, n, \{\beta\}) = (-1)^{|\beta|} \sum_{\sigma \in \alpha \sqcup\!\sqcup \beta^t} A_n^{\text{tree}}(1, \{\sigma\}, n) \;, \qquad (0.0.5)$$

  where α and β are disjoint ordered subsets of $\{2, 3, \ldots, n-1\}$ such that $\alpha \cup \beta = \{2, 3, \ldots, n-1\}$, the β^t operation reverses the order of the elements of β, $|\beta|$ is the number of elements in β and $\alpha \sqcup\!\sqcup \beta^t$ denotes those permutations of $\alpha \cup \beta$ which preserve the order of the elements in α and β^t.

Therefore, some rearrangements can be made in the general expression of Eq. (0.0.1). In particular, from the identities above, one can check explicitly that there are $(n-2)!$ independent partial amplitudes. Considering gluons to transform under the adjoint representation of $SU(N_c)$, the full amplitude can be written alternatively [11] by means of the color structure constants f^{abc} as

$$\mathcal{A}_n^{\text{tree}} = (ig)^{n-2} \sum_{\sigma \in S_{n-2}} f^{a_1 a_{\sigma_2} x_1} f^{x_1 a_{\sigma_3} x_2} \dots f^{x_{n-3} a_{\sigma_{n-1}} a_n} A_n^{\text{tree}}[1, \sigma(2, \dots, n-1), n] \ , \quad (0.0.6)$$

where now two of the external legs are fixed.

The most important advantage of this representation is that each of the terms in the sum, and consequently each of the color-ordered amplitudes, is gauge invariant. Compared to the general Feynman diagram decomposition of scattering amplitudes, it is already a remarkable step forward.

Moreover, partial amplitudes give rise to further identities such as

- Bern-Carrasco-Johansson (BCJ) relations:

$$\begin{aligned} & s_{1,n-2} A_n^{\text{tree}}(1, n-2, 2, 3, \dots, n-1, n) \\ & \qquad + \sum_{j=2}^{n-3} \sum_{k=1}^{j} s_{k,n-2} A_n^{\text{tree}}(1, 2, \dots, j, n-2, j+1, \dots, n-1, n) \\ & \qquad\qquad - s_{n-2,n} A_n^{\text{tree}}(1, 2, \dots, n-1, n-2, n) = 0 \end{aligned} \quad (0.0.7)$$

  Derived for the first time in [12], these identities reduce the number of independent partial amplitudes even further, up to $(n-3)!$.

Nevertheless, apart from just constituting a physically meaningful and gauge invariant decomposition of the full amplitude, the explicit expressions for these partial amplitudes turn out to adopt additionally a quite compact form. Taking into account the polarization dependence of the particles in the amplitude, it can be seen by induction for an arbitrary multiplicity n that the following configurations vanish

$$\begin{aligned} A_n^{\text{tree}}(1^+, \dots, n^+) = A_n^{\text{tree}}(1^-, \dots, n^-) = 0 \ , \\ A_n^{\text{tree}}(1^+, 2^+, \dots, i^-, \dots, n^+) = A_n^{\text{tree}}(1^-, 2^-, \dots, i^+, \dots, n^-) = 0 \ . \end{aligned} \quad (0.0.8)$$

Hence, the first non-trivial contribution to tree-level n-gluon amplitudes comes from $A_n^{\text{tree}}(1^+, 2^+, \dots, i^-, \dots, j^-, \dots, n^+)$. This helicity configuration in which two of the particles carry opposite helicities is known as *maximally helicity violating* (MHV) sector.

Written in terms of helicity spinors (see Appendix A.1 for details), it reads

$$A_n^{\text{tree}}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle ij\rangle^4}{\prod_{k=1}^{n}\langle k\, k+1\rangle} \,. \tag{0.0.9}$$

This formula, known as *Parke-Taylor formula*, was first obtained in [13], and settled the spinor-helicity formalism as a powerful representation for the description of the on-shell degrees of freedom of scattering amplitudes.

Based on this background for color decomposition and the onshell representation of partial subamplitudes, many novel mathematical tools were developed beyond the standard QFT approaches. For example, *onshell recursion relations*[1] allow for the recursive analytic construction of general n-point tree-level amplitudes with 3-point amplitudes as the unique building blocks; or *unitarity methods*, which allow to compute loop-level corrections from the tree-level amplitudes and an integral basis of one-loop Feynman integrals.

However, apart from all these novel techniques, we are focusing now on an intriguing symmetry of scattering amplitudes that will be used throughout the whole thesis, and establishes a simple procedure to connect gauge and gravitational amplitudes: *color-kinematics duality* [12, 15, 16]. The full tree-level n-gluon amplitude in Eq. (0.0.6) can alternatively be organized in terms of trivalent diagrams —i.e. diagrams with only cubic vertices—, leading to the following expression

$$\frac{1}{g^{n-2}}\, A_n^{\text{tree}} = \sum_{i\in\Gamma} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \,, \tag{0.0.10}$$

where the sum $\sum_{i\in\Gamma}$ runs over trivalent diagrams, denominators are given by the product of all poles contributing to each diagram, we have generically denoted color factors as c_i and all the remaining kinematic structure of the amplitude is encoded inside the numerators n_i.

Starting for example with the explicit form of Feynman diagrams, all trivalent diagrams of the amplitude would correspond to a particular term in Eq. (0.0.10) whereas all those diagrams containing 4-gluon vertices could be split and relocated by means of color factor identities among the different trivalent terms. The direct consequence is that numerators are not unambiguously defined and there is still some freedom —normally refered as *generalized gauge transformation*— to deform them and still have a gauge invariant amplitude.

In particular, taking into account that color factors fulfill some Jacobi identities, color-kinematics duality states that it is always possible to find a representation for the kinematic numerators in such a way that they mimic the same identities

$$\begin{aligned} c_i = -c_j \quad &\Leftrightarrow \quad n_i = -n_j \,, \\ c_i + c_j + c_k = 0 \quad &\Leftrightarrow \quad n_i + n_j + n_k = 0 \,, \end{aligned} \tag{0.0.11}$$

without spoiling the validity of the representation in Eq. (0.0.10).

[1] Also known as *Britto-Cachazo-Feng-Witten (BCFW) recursion relations* in the literature. See [14].

This result is remarkable because, apart from indirectly being the origin of the BCJ relations described in Eq. (0.0.7), it turns out that in such a representation, substituting the color factor c_i by a second copy of the kinematic numerator n_i gives rise to a well defined amplitude: the tree-level n-graviton amplitude of the Einstein-Hilbert theory

$$\frac{-i}{(\kappa/2)^{n-2}} M_n^{\text{tree}} = \sum_{i\in\Gamma} \frac{n_i^2}{\prod_{\alpha_i} p_{\alpha_i}^2} \,. \tag{0.0.12}$$

The procedure is called the *double-copy prescription*, and it was one of the first proposals in the field theory limit for the intriguing connection between gauge theories and gravity. Some precursors for this link in String Theory between open and closed string amplitudes go under the name of *Kawai-Lewellen-Tye (KLT) relations* [17, 18] (details will be given in Ch. 1 and 2).

Furthermore, the construction of the gravitational amplitude in Eq. (0.0.12) is more general than that. Given two different representations n_i and $\tilde{n}_i$ of the kinematic numerators, it is clear that the expression

$$\sum_{i\in\Gamma} \frac{c_i \Delta_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0 \qquad \text{for} \qquad \Delta_i \equiv n_i - \tilde{n}_i \,, \tag{0.0.13}$$

identically vanishes due to the fact that color factors fulfill the Jacobi identities. If we translate the same conditions into one of the representations n_i, then

$$\sum_{i\in\Gamma} \frac{n_i \Delta_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0 \,, \tag{0.0.14}$$

meaning that two different representations of the kinematic numerators give rise to the same n-graviton tree-level amplitude

$$\frac{-i}{(\kappa/2)^{n-2}} M_n^{\text{tree}} = \sum_{i\in\Gamma} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \,, \tag{0.0.15}$$

provided one of them satisfies an analogue of the Jacobi identities in Eq. (0.0.11).

In the recent years, many distinct[2] modern mathematical methods have arised that serve as an alternative for the study of phenomenological and theoretical aspects of scattering amplitudes. Among them, the ones described above could serve as an introduction to the topic. However, there constantly arise novel mechanisms providing different insights to the field. In particular, in this thesis we are going to focus on some peculiar mathematical structure seldom reviewed in the literature: *planar radiation zeros*. Broadly speaking, radiation zeros characterize the phase space configurations for which the full scattering amplitude of a given process vanishes. Besides having implications from the phenomenological point of view, we have proven them to be equally useful from the theoretical point

[2]Many of them are reviewed in [4–9].

of view.

In Chapters 1 and 2, we have studied their appearance in tree-level MHV amplitudes belonging to biadjoint scalar theory, Yang-Mills theory and Einstein-Hilbert gravity via color-kinematics duality, finding some interesting behaviors. Gauge zeros are found to be determined inside a “projective space” when studied in the planar limit, whereas gravitational amplitudes are found to vanish identically whenever all particle momenta lie on the same plane. The former is related to the soft behavior of the theory, as it will be discussed in detail throughout the text, and the latter is simply a consequence of symmetry arguments for a three dimensional gravity. Having studied the implications of these planar zeros, Chapter 4 serves as a complement, trying to elucidate the actual origin and their interpretation within the Cachazo-He-Yuan (CHY) formalism. This formalism is an alternative framework for the description of scattering amplitudes of massless particles relying on the so-called *scattering equations* (SE) and their exact resolution, which is by no means a trivial task. Thus, Chapter 3 is devoted to review its construction and to show how Sudakov parametrization implies a great simplification of the problem. Written as punctures on the Riemann sphere, we find that the solutions to the SE become remarkably simple functions when expressed in terms of Sudakov variables. Moreover, the amplitudes themselves adopt rather compact formulas in terms of these parameters, suggesting Sudakov parametrization to be a natural candidate for the description of the formalism.

Chapter 1

Planar Zeros in Gauge Theories and Gravity

1.1 Introduction

In the last decade, many studies have permitted a deeper understanding of the relationship between gravity and gauge theories from the point of view of scattering amplitudes (see [5] for a comprehensive review). One of the most interesting results is color-kinematics duality [12, 15], which allows the construction of gravity amplitudes by replacing color factors by a second copy of the kinematic numerators. This double copy structure has a historic antecedent in the Kawai, Lewellen, and Tye (KLT) relations [17], showing how, at tree level, closed string amplitudes admit a decomposition in terms of products of open string amplitudes. Similar structures have been found in various other setups [19–21]. It seems clear that, at the level of scattering amplitudes, there is a sense in which gravity can be considered the "square" of a gauge theory.

Given this double copy structure, a natural question to ask is how certain properties of gauge theory amplitudes translate into the gravitational side. One of these is the existence of radiation zeros [22–24]. This is a peculiar feature of certain scattering processes where one or more massless gauge bosons are radiated, consisting in the vanishing of the amplitude for certain phase space configurations. The phenomenon was first identified in processes involving gauge bosons trilinear couplings, in particular $u\bar{d} \to W^+\gamma$ [25, 26]. It has been experimentally observed both at the Tevatron [27] and LHC [28]. Their existence has been also studied in graviton photoproduction [29].

These so-called Type-I zeros appear for momentum configurations satisfying the constraint $Q_i = \kappa\, p_i \cdot k$, where k is the momentum of the gauge boson, Q_i and p_i are the charge and momenta of the other particles, and κ is a numerical constant. In Ref. [30] it was realized that zeros in the amplitude may also occur when the spatial momenta of the particles involved in the process lie on the same plane. These Type-II or planar zeros have been identified in the processes $e^+e^- \to W^+W^-\gamma$ [31] and $e^+e^- \to \tau\bar{\tau}\gamma$ [32], in both cases in the soft photon limit.

So far, the only study of planar zeros beyond the soft limit has been carried out in the interesting work [33], where the five parton amplitude in QCD was analyzed. Using the maximally helicity violating (MHV) formalism, planar zeros were found both for the $gg \to ggg$ and $q\bar{q} \to ggg$ processes. In the case of the five-gluon amplitude for general color factors the planar zero condition depends on the color quantum numbers of incoming and outgoing gluons.

The present chapter has a double aim. One is to study the conditions for the emergence of planar zeros in the five-gluon amplitude. We show that planar zeros are a "projective" property of the amplitude, in the sense that they are preserved by a simultaneous rescaling of the stereographic coordinates labeling the flight directions of the three outgoing gluons. In terms of stereographic coordinates, we find that the existence of planar zeros is determined by a cubic algebraic curve whose integer coefficients are given in terms of the color factors. In the case of $SU(N_c)$ gauge groups, we find that the casuistic of curves obtained for different color configurations gets broader as the rank N increases, starting with the case of $SU(2)$ where no physical zeros are found for external particles with well-defined color quantum numbers. Our second target consists in exploiting color-kinematics duality to study planar zeros in the gravitational case, where we find that the five-graviton amplitude vanishes whenever the process is planar. This can be understood applying the BCJ prescription to the equation determining the zeros in the gauge case. By replacing color factors with kinematic numerators satisfying color-kinematics duality, the condition for the planar zero is seen to be identically satisfied without further kinematic constraints.

The plan of the chapter is as follows. In Section 1.2 we review the calculation of the five gluon amplitude using the MHV formalism. Section 1.3 is devoted to the conditions for planar zeros in the gauge case, while in Section 1.4 we study the transformation of the loci of planar zeros under permutations of the color labels of the external gluons. In Section 1.5 the graviton amplitude is obtained using color-kinematics duality and the condition for the existence of amplitude zeros is obtained. Finally, in Section 1.6 we summarize our conclusions.

1.2 The five-gluon amplitude

In this section we revisit the construction of the five-gluon amplitude

$$g(p_1, a_1) + g(p_2, a_2) \longrightarrow g(p_3, a_3) + g(p_4, a_4) + g(p_5, a_5)\ , \tag{1.2.1}$$

where we take all momenta incoming. The tree level amplitude is computed in terms of 15 nonequivalent trivalent diagrams, leading to the expression

$$\begin{aligned}\mathcal{A}_5 = g^3 \bigg(&\frac{c_1 n_1}{s_{12}s_{45}} + \frac{c_2 n_2}{s_{23}s_{15}} + \frac{c_3 n_3}{s_{34}s_{12}} + \frac{c_4 n_4}{s_{45}s_{23}} + \frac{c_5 n_5}{s_{15}s_{34}} + \frac{c_6 n_6}{s_{14}s_{25}} + \frac{c_7 n_7}{s_{13}s_{25}} + \frac{c_8 n_8}{s_{24}s_{13}} \\ &+ \frac{c_9 n_9}{s_{35}s_{24}} + \frac{c_{10} n_{10}}{s_{14}s_{35}} + \frac{c_{11} n_{11}}{s_{15}s_{24}} + \frac{c_{12} n_{12}}{s_{12}s_{35}} + \frac{c_{13} n_{13}}{s_{23}s_{14}} + \frac{c_{14} n_{14}}{s_{25}s_{34}} + \frac{c_{15} n_{15}}{s_{13}s_{45}} \bigg),\end{aligned} \tag{1.2.2}$$

where we have introduced the kinematic invariants

$$s_{ij} = (p_i + p_j)^2 = 2\, p_i \cdot p_j, \qquad i < j. \tag{1.2.3}$$

The color factors in Eq. (1.2.2) are given by[1]

$$
\begin{aligned}
c_1 &= f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5}\,, & c_2 &= f^{a_1a_5b}f^{ba_4c}f^{ca_3a_2}\,,\\
c_3 &= f^{a_3a_4b}f^{ba_5c}f^{ca_1a_2}\,, & c_4 &= f^{a_4a_5b}f^{ba_1c}f^{ca_2a_3}\,,\\
c_5 &= f^{a_5a_1b}f^{ba_2c}f^{ca_3a_4}\,, & c_6 &= f^{a_1a_4b}f^{ba_3c}f^{ca_5a_2}\,,\\
c_7 &= f^{a_1a_3b}f^{ba_4c}f^{ca_5a_2}\,, & c_8 &= f^{a_1a_3b}f^{ba_5c}f^{ca_4a_2}\,,\\
c_9 &= f^{a_3a_5b}f^{ba_1c}f^{ca_2a_4}\,, & c_{10} &= f^{a_4a_1b}f^{ba_2c}f^{ca_3a_5}\,,\\
c_{11} &= f^{a_1a_5b}f^{ba_3c}f^{ca_4a_2}\,, & c_{12} &= f^{a_3a_5b}f^{ba_4c}f^{ca_1a_2}\,,\\
c_{13} &= f^{a_1a_4b}f^{ba_5c}f^{ca_3a_2}\,, & c_{14} &= f^{a_5a_2b}f^{ba_1c}f^{ca_3a_4}\,,\\
c_{15} &= f^{a_1a_3b}f^{ba_2c}f^{ca_4a_5}\,,
\end{aligned}
\tag{1.2.4}
$$

and satisfy nine independent Jacobi identities

$$
\begin{aligned}
&c_3 - c_5 + c_{14} = 0, && c_3 - c_1 - c_{12} = 0,\\
&c_4 - c_1 + c_{15} = 0, && c_4 + c_2 - c_{13} = 0,\\
&c_5 + c_2 - c_{11} = 0, && c_{13} - c_6 + c_{10} = 0,\\
&c_{14} - c_7 + c_6 = 0, && c_7 - c_8 + c_{15} = 0,\\
&c_8 - c_9 - c_{11} = 0, && (c_9 - c_{10} + c_{12} = 0).
\end{aligned}
\tag{1.2.5}
$$

On general grounds, the amplitude can be written in terms of color-ordered amplitudes as

$$
\mathcal{A}_5 = g^3 \sum_{\sigma\in S_4} c[1,\sigma(2,3,4,5)]A_5[1,\sigma(2,3,4,5)],
\tag{1.2.6}
$$

where the sum is over noncyclic permutations of the external legs. However, the set of color-ordered amplitudes is overcomplete, a fact expressed by the KK relations in Eq. (0.0.5). In the case of the five-point amplitude, there are 5×4 different ways of choosing a basis in the space of independent color structures TCS_5 [34]. We select one of these basis by fixing the incoming gluons (see Fig. 1.1), so the five-gluon amplitude in Eq. (1.2.2) can be re-expressed in terms of 3! color ordered amplitudes according to

$$
\mathcal{A}_5 = g^3 \sum_{\sigma\in S_3} c[1,2,\sigma(3,4,5)]A_5[1,2,\sigma(3,4,5)],
\tag{1.2.7}
$$

where the subamplitudes are explictly given in terms of the numerators n_i by

[1] Our conventions for the color factors differ from those in Refs. [5, 12, 15].

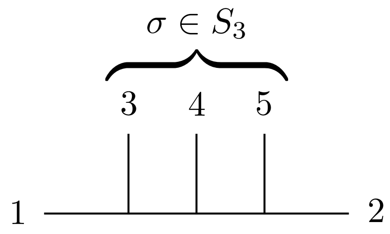


Figure 1.1: Graphic representation of the choice of subamplitudes basis in the implementation of the Kleiss-Kuijf relations.

$$\begin{aligned}
A_5[1,2,3,4,5] &= \frac{n_1}{s_{12}s_{45}} - \frac{n_2}{s_{23}s_{15}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{15}s_{34}}, \\
A_5[1,2,3,5,4] &= -\frac{n_1}{s_{12}s_{45}} - \frac{n_{13}}{s_{23}s_{14}} + \frac{n_{12}}{s_{35}s_{12}} - \frac{n_4}{s_{45}s_{23}} + \frac{n_{10}}{s_{14}s_{35}}, \\
A_5[1,2,4,3,5] &= -\frac{n_{12}}{s_{12}s_{35}} - \frac{n_{11}}{s_{24}s_{15}} - \frac{n_3}{s_{34}s_{12}} + \frac{n_9}{s_{35}s_{24}} - \frac{n_5}{s_{15}s_{34}}, \\
A_5[1,2,4,5,3] &= \frac{n_{12}}{s_{12}s_{35}} - \frac{n_8}{s_{24}s_{13}} - \frac{n_1}{s_{45}s_{12}} - \frac{n_9}{s_{35}s_{24}} - \frac{n_{15}}{s_{13}s_{45}}, \\
A_5[1,2,5,3,4] &= -\frac{n_3}{s_{12}s_{34}} - \frac{n_6}{s_{25}s_{14}} - \frac{n_{12}}{s_{35}s_{12}} + \frac{n_{14}}{s_{34}s_{25}} - \frac{n_{10}}{s_{14}s_{35}}, \\
A_5[1,2,5,4,3] &= \frac{n_3}{s_{12}s_{34}} - \frac{n_7}{s_{25}s_{13}} + \frac{n_1}{s_{12}s_{45}} - \frac{n_{14}}{s_{34}s_{25}} + \frac{n_{15}}{s_{13}s_{45}}.
\end{aligned} \tag{1.2.8}$$

Going back to the expression for the color factors in Eq. (1.2.4), these partial amplitudes are respectively associated with the six color factors c_7, c_8, c_6, c_{13}, c_{11}, and c_2.

At this point we can exploit the generalized gauge freedom in the definition of the numerators to implement color-kinematics duality, so the numerators n_i mimic the Jacobi identities (1.2.5). Solving the corresponding equations we can eliminate n_7 to n_{15} finding the following solution for the numerators in terms of the basis of color-ordered amplitudes

$$\begin{aligned}
& n_1 = -n_{12} = n_{15} = s_{12}s_{45}A_5[1,2,3,4,5], \\
& n_2 = n_3 = n_4 = n_5 = n_{11} = n_{13} = n_{14} = 0, \\
& n_6 = n_7 = n_{10} = s_{14}s_{35}A_5[1,2,3,5,4] + s_{14}(s_{35}+s_{45})A_5[1,2,3,4,5], \\
& n_8 = n_9 = s_{14}s_{35}A_5[1,2,3,5,4] + (s_{14}s_{35}+s_{14}s_{45}+s_{12}s_{45})A_5[1,2,3,4,5].
\end{aligned} \tag{1.2.9}$$

Color-kinematics duality is independent of the polarization of the gluons. Here we are going to use the MHV formalism and assign negative helicity to the incoming gluons.

Using the Parke-Taylor formula [13] we have

$$A_5[1^-,2^-,\sigma(3^+,4^+,5^+)] = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2\sigma(3)\rangle\langle\sigma(3)\sigma(4)\rangle\langle\sigma(4)\sigma(5)\rangle\langle\sigma(5)1\rangle}, \tag{1.2.10}$$

for any permutation $\sigma \in S_3$ of the last three indices. Expressing in addition the kinematic invariants in terms of spinors, $s_{ij} = \langle ij\rangle[ji]$, we arrive at the following expressions for the numerators

$$\begin{aligned}
n_1 &= -n_{12} = n_{15} = i\frac{\langle 12\rangle^4[21][54]}{\langle 23\rangle\langle 34\rangle\langle 51\rangle}, \\
n_6 &= n_7 = n_{10} = i\frac{\langle 12\rangle^4[14][52]}{\langle 23\rangle\langle 34\rangle\langle 51\rangle}, \\
n_8 &= n_9 = i\frac{\langle 12\rangle^4[24][51]}{\langle 23\rangle\langle 34\rangle\langle 51\rangle}, \\
n_2 &= n_3 = n_4 = n_5 = n_{11} = n_{13} = n_{14} = 0.
\end{aligned} \tag{1.2.11}$$

With this result, the five-gluon amplitude can be written as

$$\begin{aligned}
\mathcal{A}_5 = -ig^3\langle 12\rangle^3 \bigg(&\frac{c_2}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} + \frac{c_6}{\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle} + \frac{c_7}{\langle 25\rangle\langle 54\rangle\langle 43\rangle\langle 31\rangle} \\
&+ \frac{c_8}{\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle} + \frac{c_{11}}{\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle} + \frac{c_{13}}{\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle}\bigg).
\end{aligned} \tag{1.2.12}$$

Alternatively, this expression can be obtained from Eq. (1.2.7) by a direct application of the Parke-Taylor formula.

The spinor products appearing in the five-gluon amplitude can now be recast in terms of momenta. Working in the center-of-mass frame, the incoming momenta take the form

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1), \qquad p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1). \tag{1.2.13}$$

On the other hand, for the three outgoing gluons their spatial momenta are parametrized using stereographic coordinates $\zeta_a \in \mathbb{C}$ (with $a = 3,4,5$) according to

$$p_a = -\omega_a\left(1, \frac{\zeta_a+\overline{\zeta}_a}{1+\zeta_a\overline{\zeta}_a}, i\frac{\overline{\zeta}_a-\zeta_a}{1+\zeta_a\overline{\zeta}_a}, \frac{\zeta_a\overline{\zeta}_a-1}{1+\zeta_a\overline{\zeta}_a}\right), \tag{1.2.14}$$

where the global minus sign reflects that all momenta are taken entering the diagram. The stereographic coordinates ζ_a are related to the rapidity Y_a and the azimuthal angle ϕ_a by

$$\zeta_a = e^{Y_a+i\phi_a}. \tag{1.2.15}$$

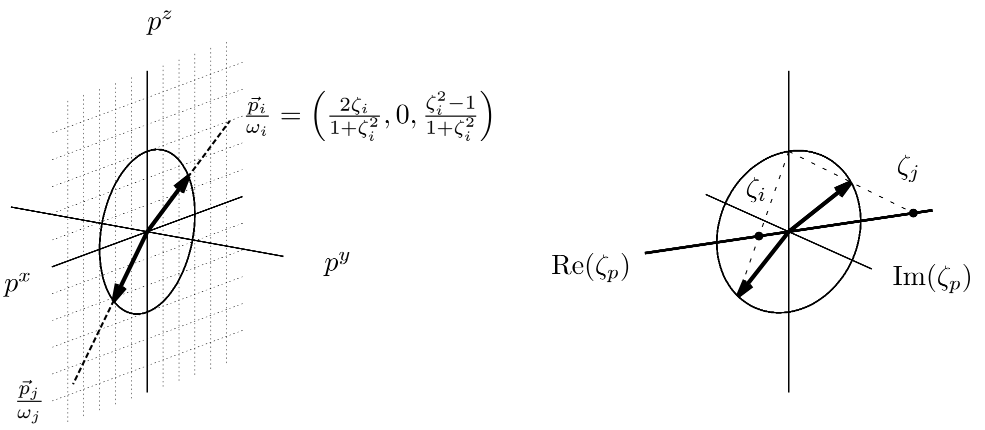


Figure 1.2: (Left) Two arbitrary unit-vectors $\frac{\mathbf{p_i}}{\omega_i}$ and $\frac{\mathbf{p_j}}{\omega_j}$ lying on the xz-plane. (Right) Stereographic projection for both of them lying on the $\mathrm{Re}(\zeta_p)$ axis.

1.3 Gauge planar zeros

We focus now on planar five-gluon scattering with general color quantum numbers. Since the incoming particles travel along the z axis, without loss of generality we can take all momenta lying on the xz-plane. This means that $p_a^y = 0$ and therefore ζ_a has to be real and the outgoing momenta read

$$p_a = -\frac{\omega_a}{1+\zeta_a^2}(1+\zeta_a^2, 2\zeta_a, 0, \zeta_a^2 - 1). \tag{1.3.1}$$

Alternatively, the planarity condition implies that all emitted particles have azimuthal angles with either $\phi_a = 0$ or $\phi_a = \pi$.

Implementing momentum conservation $p_1 + \ldots + p_5 = 0$ gives three independent equations that determine the gluon energies ω_a in terms of the center-of-mass energy $\sqrt{s}$ and the flight directions of the outgoing gluons labelled by ζ_a,

$$\begin{aligned}
\omega_3 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_3^2)(1+\zeta_4\zeta_5)}{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)}, \\
\omega_4 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_4^2)(1+\zeta_3\zeta_5)}{(\zeta_4-\zeta_3)(\zeta_4-\zeta_5)}, \\
\omega_5 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_5^2)(1+\zeta_3\zeta_4)}{(\zeta_5-\zeta_3)(\zeta_5-\zeta_4)}.
\end{aligned} \tag{1.3.2}$$

Furthermore, the finite positive energy condition $0 \leq \omega_a < \infty$ imposes constraints on the possible values of ζ_a. In particular, let us remark that finite energy implies that $\zeta_a \neq \zeta_b$ for $3 \leq a < b \leq 5$.

Using this parametrization, the amplitude (1.2.12) takes the form

$$\mathcal{A}_5 = \frac{2ig^3}{\sqrt{s}} \frac{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)}{(1+\zeta_3\zeta_4)(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)} \left[-c_2\frac{\zeta_5-\zeta_3}{\zeta_3} - c_6\frac{\zeta_4-\zeta_5}{\zeta_5} \right.$$
$$\left. +c_7\frac{\zeta_3-\zeta_5}{\zeta_5} - c_8\frac{\zeta_3-\zeta_4}{\zeta_4} + c_{11}\frac{\zeta_5-\zeta_4}{\zeta_4} + c_{13}\frac{\zeta_4-\zeta_3}{\zeta_3} \right]. \tag{1.3.3}$$

In order to find the zeros of the amplitude, we notice that the finite energy condition implies that the prefactor can never vanish. As a consequence, we find the following equation depending on the color factors

$$c_2\frac{\zeta_5-\zeta_3}{\zeta_3} + c_6\frac{\zeta_4-\zeta_5}{\zeta_5} - c_7\frac{\zeta_3-\zeta_5}{\zeta_5}$$
$$+c_8\frac{\zeta_3-\zeta_4}{\zeta_4} - c_{11}\frac{\zeta_5-\zeta_4}{\zeta_4} - c_{13}\frac{\zeta_4-\zeta_3}{\zeta_3} = 0. \tag{1.3.4}$$

The planar zero condition just derived is a homogeneous equation of vanishing degree. Since the amplitude (1.3.3) diverges whenever any of the ζ_a vanishes, we can multiply the previous equation by $\zeta_3\zeta_4\zeta_5$ without generating spurious solutions in the physical region. Taking projective coordinates

$$(\zeta_3,\zeta_4,\zeta_5) = \lambda(1,U,V), \qquad \lambda, U, V \neq 0 \tag{1.3.5}$$

the planar zeros of the five-gluon amplitude are determined by the loci defined by the following equation

$$c_7U - c_8V - c_6U^2 + c_{11}V^2$$
$$+ (c_2+c_6-c_7+c_8-c_{11}-c_{13})UV + c_{13}U^2V - c_2UV^2 = 0. \tag{1.3.6}$$

Moreover, this equation is homogeneous in the color factors and therefore independent of the normalization of the gauge group generators. Since there exists a normalization of the generators that makes all structure constants integer numbers [35], the planar zeros are determined by a cubic curve with integer coefficients.

In terms of the projective coordinates (1.3.5), the energies of the outgoing particles take the form

$$\begin{aligned}
\omega_3 &= \frac{\sqrt{s}}{2}\frac{(1+\lambda^2)(1+\lambda^2UV)}{\lambda^2(1-U)(1-V)}, \\
\omega_4 &= \frac{\sqrt{s}}{2}\frac{(1+\lambda^2U^2)(1+\lambda^2V)}{\lambda^2(U-1)(U-V)}, \\
\omega_5 &= \frac{\sqrt{s}}{2}\frac{(1+\lambda^2V^2)(1+\lambda^2U)}{\lambda^2(V-1)(V-U)}.
\end{aligned} \tag{1.3.7}$$

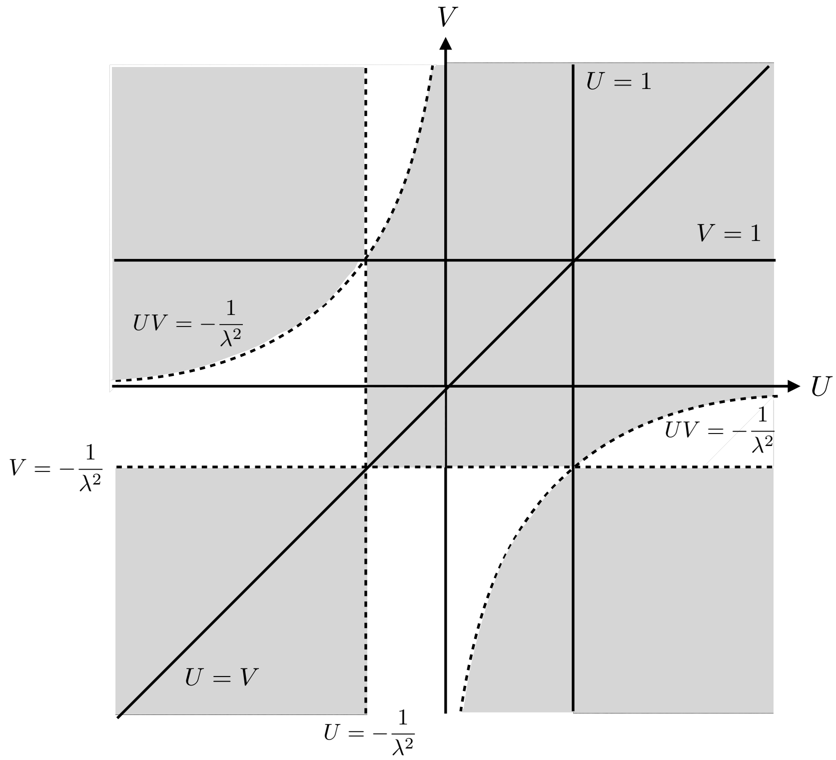


Figure 1.3: Physical regions in the UV plane for a given value of λ. The shadowed regions are unphysical points where the energy is negative for at least one of the outgoing gluons. Dashed lines correspond to the soft limits in which one or several energies tend to zero. Solid lines, including the axes, represent also unphysical points.

We have seen already that in order to keep the amplitude finite we have to exclude $U = 0$ and $V = 0$ from the physical region. Now, energy finiteness further demands that $U \neq 1$, $V \neq 1$, and $U \neq V$. By requiring $\omega_a \geq 0$ we find that, for example, the region $U > 0$, $V > 0$ has to be considered unphysical as well. Indeed, if this is the case all three numerators in (1.3.7) are positive whereas the three denominators cannot have the same sign simultaneously. As a consequence, at least one of the energies has to be negative and the configuration is excluded. Studying the values of U and V in which the three energies are simultaneously positive for a given λ, we arrive at the physical regions shown in Fig. 1.3. Notice that the plot is symmetric under the exchange $U \leftrightarrow V$.

The conformal structure of the equation defining the planar zeros indicates that each solution of Eq. (1.3.6) can be realized in infinitely many physical setups, depending on the value of the parameter λ. Notice that the boundaries of the allowed regions depend on λ as well, so they move as this parameter varies, while the position of the zeros, being a projective invariant, remain fixed.

A particularly interesting regime is the soft limit, in which one or various of the emitted gluon energies tend to zero. From Eq. (1.3.7) we see that the points in the (U, V) plane for which ω_a vanish are given by

$$\begin{aligned} UV &= -\frac{1}{\lambda^2} \qquad (\omega_3 = 0), \\ V &= -\frac{1}{\lambda^2} \qquad (\omega_4 = 0), \\ U &= -\frac{1}{\lambda^2} \qquad (\omega_5 = 0), \end{aligned} \tag{1.3.8}$$

which are indicated by the dashed lines in Fig. 1.3. On general grounds, a planar zero corresponding to a point of the cubic (1.3.6) can be physically captured in the soft limit provided there is a value of λ for which this point collides against any of the "soft" lines defining the boundaries of the physical region.

The first example to analyze is the case of incoming gluons in a singlet state for arbitrary gauge group, already studied in [33]. Using the fact that $f^{da_3b}f^{ba_4c}f^{ca_5d} \sim f^{a_3a_4a_5}$, we find

$$c_2 = c_6 = -c_7 = c_8 = -c_{11} = -c_{13} = -f^{a_3a_4a_5}. \tag{1.3.9}$$

The cubic equation then reads

$$U + V + U^2 + V^2 - 6UV + U^2V + UV^2 = 0. \tag{1.3.10}$$

The associated algebraic curve is represented in Fig. 1.4. Comparing with Fig. 1.3 we see that for small enough λ there is indeed a large part of the curve lying on physically allowed regions. In particular, for $\lambda < 1$ there are solutions with arbitrarily large $|U|$ and $|V|$.

We study next the loci defined by Eq. (1.3.6) for $SU(N)$ gauge groups with different ranks and various color configurations:

SU(2). In the case of $SU(2)$ it is easy to write a closed expression for the color factors

$$\begin{aligned} c_2 &= \delta^{a_3a_4}\epsilon^{a_2a_5a_1} - \delta^{a_2a_4}\epsilon^{a_3a_5a_1}, \\ c_6 &= \delta^{a_5a_3}\epsilon^{a_2a_4a_1} - \delta^{a_2a_3}\epsilon^{a_5a_4a_1}, \\ c_7 &= \delta^{a_1a_4}\epsilon^{a_3a_5a_2} - \delta^{a_3a_4}\epsilon^{a_1a_5a_2}, \\ c_8 &= \delta^{a_2a_5}\epsilon^{a_4a_1a_3} - \delta^{a_4a_5}\epsilon^{a_2a_1a_3}, \\ c_{11} &= \delta^{a_4a_3}\epsilon^{a_2a_5a_1} - \delta^{a_2a_3}\epsilon^{a_4a_5a_1}, \\ c_{13} &= \delta^{a_4a_5}\epsilon^{a_1a_2a_3} - \delta^{a_1a_5}\epsilon^{a_4a_2a_3}, \end{aligned} \tag{1.3.11}$$

where a convenient normalization of the gauge group generators has been chosen. In principle, the color factors can only take the values $0, \pm 1$, and ± 2, since each term on the

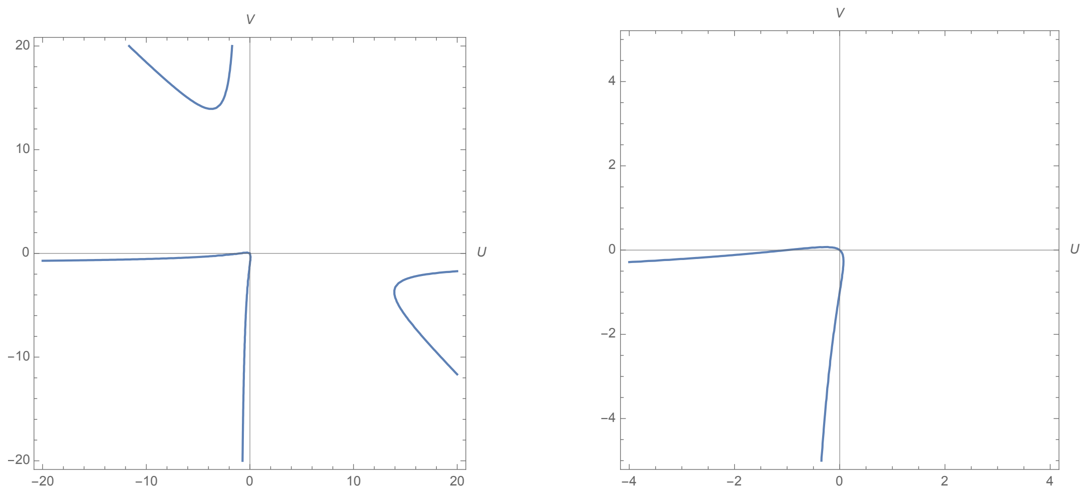


Figure 1.4: Algebraic curve associated with the planar zeros for incoming gluons in a singlet state, Eq. (1.3.10). The right panel shows a blowup of the region around $(U, V) = (0, 0)$.

right-hand side of these equations is either 0 or ± 1. However, the case ± 2 is excluded. The reason is that due to the structure of indices of the Levi-Civita tensor, sharing the last two entries, they cannot have oposite signs. As a consequence, they cannot add up and we conclude that for $SU(2)$ the color factors satisfy $c_i = 0, \pm 1$.

An exploration of the possible external color numbers show that there are no solutions containing physical points. We illustrate this with a few examples. Our first case has color structure $(a_1, a_2, a_3, a_4, a_5) = (2, 3, 1, 1, 1)$, giving the same value for all color factors

$$c_2 = c_6 = c_7 = c_8 = c_{11} = c_{13} = 1. \tag{1.3.12}$$

The resulting cubic equation completely factorizes as

$$(U - 1)(V - 1)(U - V) = 0. \tag{1.3.13}$$

We see that the three solutions lie outside the physical region and as a consequence there are no planar zeros for this gauge configuration.

Next we try $(a_1, a_2, a_3, a_4, a_5) = (2, 2, 2, 1, 3)$, which corresponds to color factors

$$c_2 = c_7 = c_8 = c_{13} = 0, \qquad c_6 = -c_{11} = 1. \tag{1.3.14}$$

In this case the equation for the zeros becomes quadratic and factorizes as

$$(U - V)^2 = 0. \tag{1.3.15}$$

The geometric loci of zeros coincide again with the unphysical region corresponding to two particles in the final state with infinite energy.

As a last example, we take $(a_1, a_2, a_3, a_4, a_5) = (1, 2, 2, 2, 3)$, which gives

$$c_2 = c_8 = c_{11} = c_{13} = 0, \qquad c_6 = c_7 = 1. \tag{1.3.16}$$

In this case the cubic again degenerates into a quadratic equation

$$U(U-1) = 0, \tag{1.3.17}$$

which has no physical solutions.

To summarize, a scan of possible values of the external color numbers show that the only curves obtained in this case coincide with unphysical regions in the plot of Fig. 1.3, $U = 0, 1$, $V = 0, 1$ or $U = V$. The only possibility for planar zeros in this case is to consider external states without well-defined color numbers, such as the singlet case studied above.

SU(3). We work out a first example where we take color quantum numbers $(a_1, a_2, a_3, a_4, a_5) = (7, 7, 6, 1, 5)$ and color factors

$$c_2 = -c_7 = c_8 = -c_{13} = 2, \qquad c_6 = -c_{11} = -1. \tag{1.3.18}$$

The planar zeros are given by the factorized cubic

$$(U + V - 2)(U + V - 2UV) = 0. \tag{1.3.19}$$

This is a hyperbola together with its tangent at $(U, V) = (1, 1)$ (see the left panel of Fig. 1.5). The loci has nonvanishing intersection with the physically allowed region in the UV plane for appropriate values of λ.

A different hyperbola is obtained for $(a_1, a_2, a_3, a_4, a_5) = (1, 4, 1, 2, 6)$ with

$$c_2 = -c_{11} = -1, \qquad c_6 = -4, \qquad c_7 = c_8 = 0, \qquad c_{13} = -2. \tag{1.3.20}$$

The equation determining the zeros also factorizes in this case, giving

$$(2U - V)(-2U + V + UV) = 0. \tag{1.3.21}$$

Again, we have a hyperbola and one of its tangents, this time at the origin. The curves are shown in the RHS panel of Fig. 1.5.

As in the $SU(2)$ cases all examples explored for the gauge group $SU(3)$ show factorization of the cubic equation. In this latter case, however, not only the type of curves is enlarged to include hyperbolas which were absent for $SU(2)$, but the curves contain physical points. In addition, considering quantum numbers in a $SU(2)$ subgroup of $SU(3)$ generates the curves obtained for the former group.

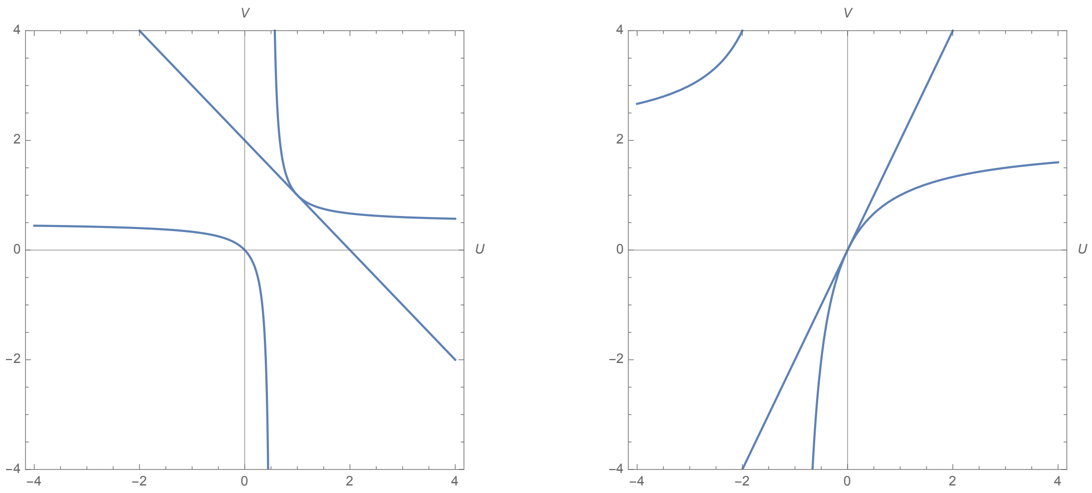


Figure 1.5: Left panel: curve (1.3.19) giving the planar zeros for the $SU(3)$ five-gluon amplitude with $(a_1, a_2, a_3, a_4, a_5) = (7, 7, 6, 1, 5)$. Right panel: the same for $(a_1, a_2, a_3, a_4, a_5) = (1, 4, 1, 2, 6)$, corresponding to Eq. (1.3.21).

SU(5). Enlarging the gauge group to $SU(5)$ brings more general types of cubic algebraic curves. This is for example the case taking $(a_1, a_2, a_3, a_4, a_5) = (17, 19, 19, 18, 23)$. The resulting color factors are

$$c_2 = c_{13} = 0, \qquad c_6 = c_8 = 2, \qquad c_7 = c_{11} = 1. \tag{1.3.22}$$

Since c_2 and c_{13} vanish, it results in the following quadratic equation determining the planar zeros

$$U - 2U^2 - 2V + 2UV + V^2 = 0. \tag{1.3.23}$$

Unlike the examples encountered for $SU(2)$ and $SU(3)$, this curve does not factorize and corresponds to the hyperbola shown in the LHS panel of Fig. 1.6.

A second interesting example is provided by $(a_1, a_2, a_3, a_4, a_5) = (19, 18, 23, 17, 19)$. The corresponding color factors are

$$c_2 = c_{11} = 0, \qquad c_6 = c_8 = 2, \qquad c_7 = c_{13} = 1. \tag{1.3.24}$$

The resulting equation for the zero

$$U - 2U^2 - 2V + 2UV + U^2V = 0 \tag{1.3.25}$$

is the cubic curve shown in the RHS panel of Fig. 1.6.

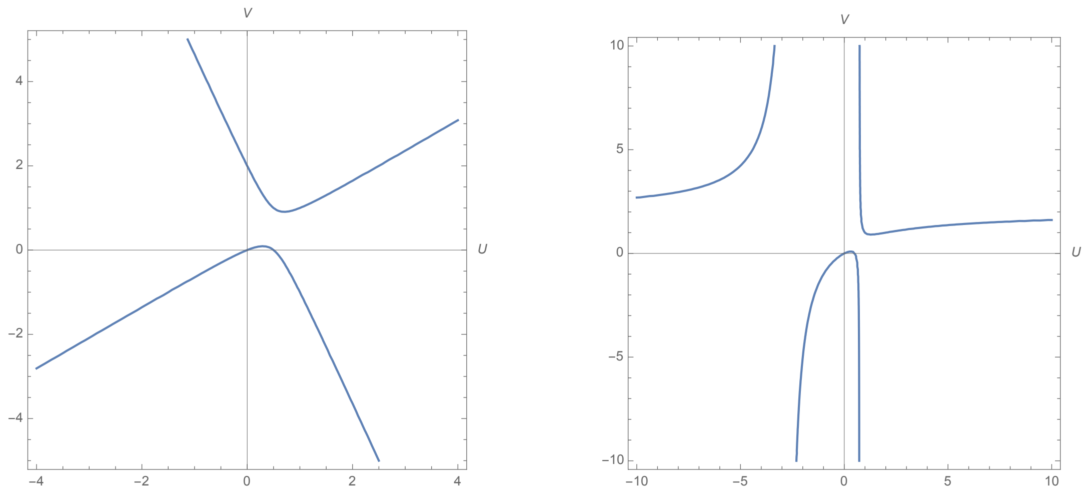


Figure 1.6: Two examples of planar zero curves for the group $SU(5)$: the hyperbola in the LHS panel corresponds to $(a_1, a_2, a_3, a_4, a_5) = (17, 19, 19, 18, 23)$, resulting in Eq. (1.3.23). Equation (1.3.25) is represented on the RHS panel, corresponding to color indices $(a_1, a_2, a_3, a_4, a_5) = (19, 18, 23, 17, 19)$.

As a last example we take $(a_1, a_2, a_3, a_4, a_5) = (19, 19, 18, 23, 17)$, with color factors

$$c_2 = -c_7 = -2, \qquad c_6 = c_8 = -c_{11} = -c_{13} = 1. \tag{1.3.26}$$

We get the cubic curve

$$2U - U^2 - V - U^2V - V^2 + 2UV^2 = 0, \tag{1.3.27}$$

which, as shown in Fig. 1.7, contains a singular point at $(U, V) = (1, 1)$.

We see how $SU(5)$ provides more general types of curves than the ones found for unitary groups of lower rank. We also have to take into account that $SU(5)$ contains $SU(3)$ and $SU(2)$ subgroups. Using the standard generators (see, for example, [36]) these subgroups are respectively generated by $\{T^1, \ldots, T^8\}$ and $\{T^{21}, T^{22}, T^{23}\}$. Thus, setting the external indices in the subsets $(1, \ldots, 8)$ or $(21, 22, 23)$ we recover previous examples. For instance, $(a_1, a_2, a_3, a_4, a_5) = (7, 7, 6, 1, 5)$ gives the curve shown in the LHS panel of Fig. 1.5, whereas $(a_1, a_2, a_3, a_4, a_5) = (22, 23, 21, 21, 21)$ reproduces Eq. (1.3.13).

1.4 Planar zeros and color permutations

It is interesting to see how the zeros here investigated transform under permutations of the color quantum numbers of the external particles. We begin considering those permutations preserving the choice of amplitudes basis implied by Eq. (1.2.7). These are elements of S_3 permuting the color indices of the three outgoing gluons (see Fig. 1.1).

In order to find the action of these permutations on the geometric loci of planar zeros,

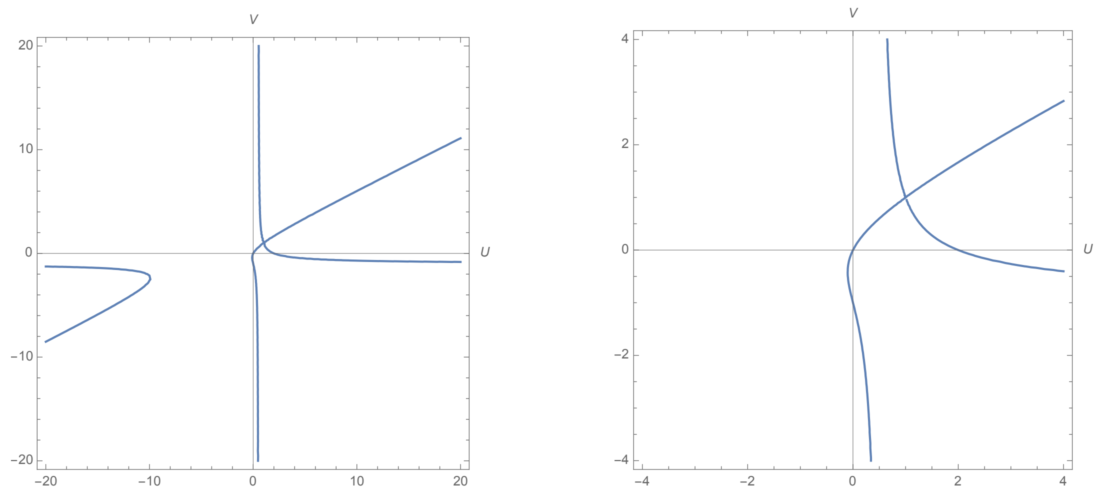


Figure 1.7: Singular curve in Eq. (1.3.27) corresponding to $(a_1, a_2, a_3, a_4, a_5) = (19, 19, 18, 23, 17)$. The RHS panel blows up the region around the origin.

we can see how the color factors (1.2.4) transform under permutations of the (a_3, a_4, a_5) color indices. Here instead we use a more geometric approach and work with the homogenization of the cubic equation (1.3.6)

$$c_7 Z^2 U - c_8 Z^2 V - c_6 Z U^2 + c_{11} Z V^2 + (c_2 + c_6 - c_7 + c_8 - c_{11} - c_{13}) ZUV \\ + c_{13} U^2 V - c_2 U V^2 = 0. \qquad (1.4.1)$$

The group S_3 acts passively by permutation of the coordinates (Z, U, V). Let us discuss the geometrical meaning of these transformations. Equation (1.4.1) is defined in the whole projective plane, which is covered by the three affine patches centered at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The group S_3 is generated by S_2 transformations interchanging the two coordinates within each patch, together with cyclic permutations of the three patches. This defines a coset decomposition of S_3 with respect to its cyclic subgroup generated by (123).

The physically allowed regions shown in Fig. 1.3 correspond to the patch centered at $(1, 0, 0)$. It has been already pointed out that it is invariant under the interchange of the two coordinates $U \leftrightarrow V$. Moreover, the corresponding plots in the other two affine coordinate patches are identical to this one. This is easy to see from Eq. (1.3.2), where it is glaring that cyclic permutations of the three patches only interchange the energies of the three outgoing gluons. Thus, the positivity conditions remain algebraically the same in any of the three affine patches. The final conclusion is that S_3 only acts on the axes labels of the plot in Fig. 1.3. This is a passive version of the fact that the energies of the outgoing gluons are determined by momentum conservation alone and that the color structures play no role in it.

Applying a permutations of S_3 to Eq. (1.4.1), we find that the color factors transform under the (six-dimensional) regular representation of the group: in particular, if we write

$\boldsymbol{C} = (c_2, c_6, c_7, c_8, c_{11}, c_{13})^T$ the group S_3 acts through the matrices

$$
\begin{aligned}
(1)(2)(3) &= \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, &
(123) &= \begin{pmatrix} 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0 \end{pmatrix}, \\
(132) &= \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix}, &
(12)(3) &= \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}, \\
(13)(2) &= \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0 \end{pmatrix}, &
(1)(23) &= \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}.
\end{aligned}
\tag{1.4.2}
$$

Notice that the combination of color factors in the coefficient of the ZUV term itself transforms with the one-dimensional parity representation,

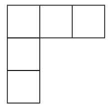
$$\sigma(c_2 + c_6 - c_7 + c_8 - c_{11} - c_{13}) = (-1)^{\pi(\sigma)}(c_2 + c_6 - c_7 + c_8 - c_{11} - c_{13}), \tag{1.4.3}$$

where $\pi(\sigma) = 0, 1$ for even and odd permutations respectively. This is a consequence of the fact that ZUV is an invariant under the permutation group.

Applying the transformations given by the matrices (1.4.2) to the curve (1.3.10) obtained in the case of the scattering of two gluons in a singlet state, $\boldsymbol{C} = (1, 1, -1, 1, -1, -1)^T$, we find that it is invariant, since $\boldsymbol{C}$ transforms with the parity of the permutation, $\sigma(\boldsymbol{C}) = (-1)^{\pi(\sigma)}\boldsymbol{C}$. This means the curve shown in the plot in Fig. 1.4 describes the planar zeros in all three coordinate patches. Incidentally, notice that the curve is invariant as well under the interchange of the two coordinates in any of the three corresponding plots.

The curves presented in Section 1.3 are expressed in the patch $(1, 0, 0)$. Under permutation of the two coordinates, some solutions remain invariant, such as Eq. (1.3.19), or get mapped into a different solution.

We can also consider the transformation of the curves with respect to general color permutations. They form the group TCS_5 which is identified with the cyclic Lie operad $\mathsf{Lie}((5))$. Its structure has been studied in Ref. [34], where it was found that its action on the six independent color structures is given by the following six-dimensional representation of S_5



.

Unlike the transformations of S_3 studied above, those in $\mathsf{TCS}_5\backslash S_3$ do not act on the geometric loci of planar zeros by permutation of the projective coodinates (Z, U, V). The most obvious example is provided by the interchange of the color indices of the incoming gluons a_1, a_2. From Eq. (1.2.4) we find that this transformation acts linearly on $\boldsymbol{C}$ through the matrix

$$(12)(3)(4)(5) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \tag{1.4.4}$$

where we have used the cycle notation for the elements of S_5. It is interesting to point out that this transformation leaves invariant the coefficient of the ZUV coefficient in (1.4.1). This last property is not shared by other transformations in $\mathsf{TCS}_5\backslash S_3$. For example,

$$(134)(25) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}, \quad (1245)(3) = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}, \tag{1.4.5}$$

act on $c_2 + c_6 - c_7 + c_8 - c_{11} - c_{13}$ respectively as

$$\begin{aligned} (134)(25)(c_2 + c_6 - c_7 + c_8 - c_{11} - c_{13}) &= c_2 - c_6 + c_7 - c_8 - c_{11} - c_{13}, \\ (1245)(3)(c_2 + c_6 - c_7 + c_8 - c_{11} - c_{13}) &= c_2 - c_6 - c_7 + c_8 - c_{11} + c_{13}. \end{aligned} \tag{1.4.6}$$

Through its transformations of the color factors, TCS_5 acts on the curves determining the planar zeros. In fact, this action can be used to generate the whole orbit of projective curves associated with the permutations of the color quantum numbers of the interacting gluons.

1.5 Graviton planar zeros from color-kinematics duality

We turn now to the problem of planar zeros in the five-graviton tree level amplitude. The gravitational amplitude can be constructed from its gluon counterpart (1.2.2) using the

BCJ prescription [12, 15],

$$-i\mathcal{M}_5 = \left(\frac{\kappa}{2}\right)^3 \left(\frac{n_1^2}{s_{12}s_{45}} + \frac{n_2^2}{s_{23}s_{15}} + \frac{n_3^2}{s_{34}s_{12}} + \frac{n_4^2}{s_{45}s_{23}} + \frac{n_5^2}{s_{15}s_{34}} + \frac{n_6^2}{s_{14}s_{25}} + \frac{n_7^2}{s_{13}s_{25}} \right.$$
$$\left. + \frac{n_8^2}{s_{24}s_{13}} + \frac{n_9^2}{s_{35}s_{24}} + \frac{n_{10}^2}{s_{14}s_{35}} + \frac{n_{11}^2}{s_{15}s_{24}} + \frac{n_{12}^2}{s_{12}s_{35}} + \frac{n_{13}^2}{s_{23}s_{14}} + \frac{n_{14}^2}{s_{25}s_{34}} + \frac{n_{15}^2}{s_{13}s_{45}}\right), \tag{1.5.1}$$

provided the numerators n_i satisfy color-kinematics duality, with κ the gravitational coupling. Taking the graviton polarizations $(1^-, 2^-, 3^+, 4^+, 5^+)$, we use our expression for the MHV gauge amplitude given in Eq. (1.2.12) to write

$$-i\mathcal{M}_5 = -i\left(\frac{\kappa}{2}\right)^3 \langle 12\rangle^3 \left(\frac{n_2}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} + \frac{n_6}{\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle} + \frac{n_7}{\langle 25\rangle\langle 53\rangle\langle 43\rangle\langle 31\rangle}\right.$$
$$\left. + \frac{n_8}{\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle} + \frac{n_{11}}{\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle} + \frac{n_{13}}{\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle}\right). \tag{1.5.2}$$

Using the form of the gauge theory numerators in Eq. (1.2.11), after a bit of algebra we arrive at the simpler expression [37]

$$-i\mathcal{M}_5 = -\left(\frac{\kappa}{2}\right)^3 \frac{\langle 12\rangle^7[41][52]}{\langle 12\rangle\langle 14\rangle\langle 23\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle}\left(1 - \frac{\langle 14\rangle\langle 25\rangle[42][51]}{\langle 15\rangle\langle 24\rangle[41][52]}\right). \tag{1.5.3}$$

The term inside the parenthesis can be further simplified taking into account the relation $s_{ij} = \langle ij\rangle[ji]$,

$$1 - \frac{\langle 14\rangle\langle 25\rangle[42][51]}{\langle 15\rangle\langle 24\rangle[41][52]} = 1 - \left(\frac{\langle 14\rangle\langle 25\rangle}{\langle 15\rangle\langle 24\rangle}\right)^2\left(\frac{s_{15}s_{24}}{s_{14}s_{25}}\right), \tag{1.5.4}$$

which in turn can be written as a function of $\Delta\phi_{45}$, the difference of azimuthal angles of particles 4 and 5 (see the Appendix of Ref. [33]),

$$\left(\frac{\langle 14\rangle\langle 25\rangle}{\langle 15\rangle\langle 24\rangle}\right)^2\left(\frac{s_{15}s_{24}}{s_{14}s_{25}}\right) = e^{2i\Delta\phi_{45}}. \tag{1.5.5}$$

With this, the five-graviton tree level amplitude reads

$$-i\mathcal{M}_5 = -\left(\frac{\kappa}{2}\right)^3 \frac{\langle 12\rangle^7[41][52]}{\langle 12\rangle\langle 14\rangle\langle 23\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle}\left(1 - e^{2i\Delta\phi_{45}}\right). \tag{1.5.6}$$

Given our choice of reference frame, planarity implies that for any two outgoing momenta their azimuthal angles must satisfy $\Delta\phi_{ij} = 0, \pi$. In both cases we find from (1.5.6) that the gravitational amplitude vanishes

$$-i\mathcal{M}_5\Big|_{\text{planar}} = 0. \tag{1.5.7}$$

Unlike the gauge case where the cancellation condition depends on the color factors of the incoming particles, the graviton amplitude automatically vanishes in the limit of planar scattering.

We now show that this is a consequence of color kinematic duality. In gauge theories we have seen that the vanishing of the amplitude in the planar case leads to a nontrivial condition on the momenta of the outgoing particles given in Eq. (1.3.4). In fact, the numerators in Eq. (1.2.11) have been chosen to satisfy color-kinematics duality, so we can obtain the planar zero condition for gravity by replacing the color factors $\{c_2, c_6, c_7, c_8, c_{11}, c_{13}\}$ in (1.3.4) with the corresponding numerators $\{n_2, n_6, n_7, n_8, n_{11}, n_{13}\}$. In terms of the stereographic coordinates, the latter are given by

$$\begin{aligned}
n_6 &= n_7 = is^{\frac{3}{2}}\frac{(\zeta_3-\zeta_5)\zeta_5}{\zeta_3(1+\zeta_4\zeta_5)},\\
n_8 &= is^{\frac{3}{2}}\frac{(\zeta_3-\zeta_5)\zeta_4}{\zeta_3(1+\zeta_4\zeta_5)},\\
n_2 &= n_{11} = n_{13} = 0.
\end{aligned} \tag{1.5.8}$$

After this substitution, the condition for the existence of a zero in the amplitude is identically satisfied

$$\begin{aligned}
-n_2\frac{\zeta_5-\zeta_3}{\zeta_3} &- n_6\frac{\zeta_4-\zeta_5}{\zeta_5} + n_7\frac{\zeta_3-\zeta_5}{\zeta_5} - n_8\frac{\zeta_3-\zeta_4}{\zeta_4} + n_{11}\frac{\zeta_5-\zeta_4}{\zeta_4} + n_{13}\frac{\zeta_4-\zeta_3}{\zeta_3}\\
&= is^{\frac{3}{2}}\frac{(\zeta_3-\zeta_5)}{\zeta_3(1+\zeta_4\zeta_5)}\Big(-\zeta_4+\zeta_5+\zeta_3-\zeta_5-\zeta_3+\zeta_4\Big) = 0.
\end{aligned} \tag{1.5.9}$$

This implies that, in the gravitational case, planarity is enough to make the amplitude vanish, without additional kinematic conditions to be satisfied by the stereographic coordinates of the outgoing gravitons.

1.6 Closing remarks

We have studied the presence of planar zeros in both Yang-Mills theories and gravity. For the case of gauge theories, we have represented in Fig. 1.8 the kinematics on the interaction plane for two typical planar zeros within the same color configuration. By varying the value of $\lambda \equiv \zeta_3$ while keeping $U \equiv \zeta_4/\zeta_3$ and $V \equiv \zeta_5/\zeta_3$ constant, these processes can be deformed into a different one with the emission, for example, of one or more soft gluons while the total amplitude remains equal to zero. This happens because planar zeros live in the projective U-V plane and are therefore invariant under a simultaneous rescaling of the three outgoing stereographic coordinates.

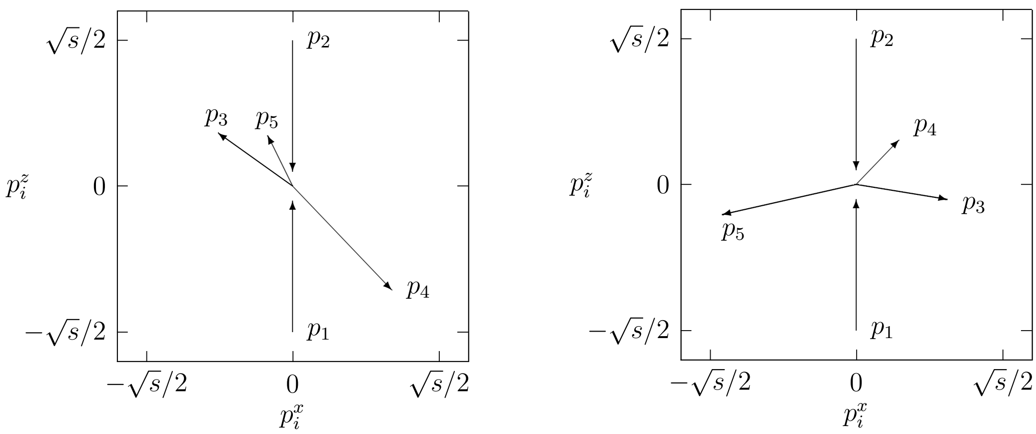


Figure 1.8: Graphic representation of the kinematic configuration on the interaction plane for two planar zeros. In both cases the gauge group is SU(3) with color configuration $(7, 7, 6, 1, 5)$, so they correspond to two points on the curve shown on the left panel of 1.5. The left panel represents the solution $(\zeta_3, \zeta_4, \zeta_5) = (-1.95, 0.4, -4.3)$, whereas in the right panel $(\zeta_3, \zeta_4, \zeta_5) = (0.85, 2.5, -0.8)$. In the projective coordinates (1.3.5), they correspond to $\lambda = -1.95$, $U = -0.21$, $V = 2.21$ and $\lambda = 0.85$, $U = 2.94$, $V = -0.94$ respectively.

Without loss of generality we considered the situation in which the scattering takes place in the $y = 0$ plane. Planar zeros on a different interaction plane can be obtained by applying rotations to the solutions studied here. In particular, the Lorentz group acts on the stereographic coordinates parametrizing the direction of the momenta through $SL(2, \mathbb{C})$ transformations [38]

$$\zeta_k' = \frac{a\zeta_k + b}{c\zeta_k + d}, \qquad ad - bc = 1, \tag{1.6.1}$$

where for the incoming particles we have $\zeta_1 = \infty$ and $\zeta_2 = 0$. Rotations can be spotted by looking for transformations leaving invariant the energies (1.3.2), together with those of the incoming particles. They are given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \xi & -\sqrt{1-\xi^2} \\ \sqrt{1-\xi^2} & \xi \end{pmatrix}. \tag{1.6.2}$$

For real $|\xi| \leq 1$, we parametrize $\xi = \cos\phi$. This corresponds to a rotation of the interaction plane of angle 2ϕ with respect to the x-axis. Alternatively, for $|\xi| > 1$, setting $|\xi| = \cosh\chi$ the transformation implements a rotation of angle $\sin\phi' = \tanh 2\chi$ around the y-axis.

With the results here presented we have shed some light on the origin of the planar zeroes present in Yang-Mills scattering amplitudes. Our results can be generalized to an arbitrary number of external legs at Born level as we will see in Chapter 2. It will be worth further investigating the effect of quantum corrections. We have also connected, via the BCJ duality, these zeroes to the corresponding ones in gravity. In the next chapter, we will additionally study how this picture is modified when the scattering of open and closed strings is considered.

Chapter 2

Projectivity of Planar Zeros in Field and String Theory Amplitudes

2.1 Introduction

Zeros in scattering amplitudes are useful devices to test interesting properties of the standard model. For example, the vanishing of the tree-level amplitude of certain processes involving the emission of a gauge boson is very sensitive to the form of the trilinear couplings. Thus, the detection of amplitude zeros were proposed as a way to constraint the existence of anomalous couplings in the standard model [25, 26] (see [22–24] for reviews). Although these so-called type-I zeros are corrected by both loops and higher-order emissions, they manifest themselves in the existence of dips for a set of observables, a fact that has been confirmed by various experimental groups [27, 28].

A second class of amplitude zeros appear for particular kinematic configurations in which all momenta are confined to a plane [30–32]. The phenomenological implications of these planar (or type-II) zeros has been recently studied in [33] in the context of a five parton amplitude, and it was shown how the planar zeros are determined by simple relations involving rapidity differences.

In the previous chapter, we have studied the mathematical structure of planar zeros in gauge theories and gravity, focusing on the five-point amplitude for gluons and gravitons. There it was found that, once the outgoing momenta are expressed in terms of stereographic coordinates, the loci of planar zeros is determined by a cubic integer curve in the projective plane defined by these coordinates.

Although the analysis presented in Chapter 1 focused on the five-point scattering amplitude, the projective nature of the planar zeros in Yang-Mills theories is present for any multiplicity. To see this let us recall that, in a (super) Yang-Mills theory, the n-gluon tree-level amplitude can be written in the form [11]

$$\mathcal{A}_n = (ig)^{n-2} \sum_{\sigma \in S_{n-2}} c_\sigma A_n\Big(1, 2, \sigma(3\ldots, n)\Big), \tag{2.1.1}$$

where $A_n(1, \ldots, n)$ is the color-ordered amplitude and the color factors c_σ are defined in terms of the structure contants by

$$c_\sigma = f^{a_1 a_2 c_1} f^{c_1 a_{\sigma(3)} c_2} \dots f^{c_{n-3} a_{\sigma(n-1)} a_{\sigma(n)}}. \tag{2.1.2}$$

To evaluate the amplitude in the planar limit, it is convenient to consider a center-of-mass frame where the incoming particles propagate along the z axis:

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1)\,, \qquad\qquad p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1)\,, \tag{2.1.3}$$

while the momenta of the on-shell outgoing gluons can be parametrized in terms of stereographic coordinates according to

$$p_a = -\omega_a\left(1, \frac{\zeta_a+\overline{\zeta}_a}{1+\zeta_a\overline{\zeta}_a}, i\frac{\overline{\zeta}_a-\zeta_a}{1+\zeta_a\overline{\zeta}_a}, \frac{\zeta_a\overline{\zeta}_a-1}{1+\zeta_a\overline{\zeta}_a}\right) \quad \text{with} \quad a=3,\dots,n. \tag{2.1.4}$$

For the particular case of MHV amplitudes, the Parke-Taylor formula [13] gives the following expression for the color-ordered amplitudes

$$A_n\Big(1^-,2^-,\sigma(3^+,\dots,n^+)\Big) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2\sigma(3)\rangle\dots\langle\sigma(n-1)\sigma(n)\rangle\langle\sigma(n)1\rangle}. \tag{2.1.5}$$

Without loss of generality, we can consider planar scattering where all momenta lie on the plane $y=0$, which means that the stereographic coordinates giving the direction of flight of the outgoing gluons are all real ($\zeta_a = \overline{\zeta}_a$). The relevant spinor inner products are computed to be

$$\begin{aligned}
\langle 12\rangle &= \sqrt{s},\\
\langle 1\sigma(j)\rangle &= -i\sqrt{2}s^{\frac{1}{4}}\frac{|\zeta_{\sigma(j)}|}{\zeta_{\sigma(j)}}\sqrt{\frac{\omega_{\sigma(j)}}{1+\zeta^2_{\sigma(j)}}},\\
\langle 2\sigma(j)\rangle &= -i\sqrt{2}s^{\frac{1}{4}}|\zeta_{\sigma(j)}|\sqrt{\frac{\omega_{\sigma(j)}}{1+\zeta^2_{\sigma(j)}}},\\
\langle\sigma(j)\sigma(k)\rangle &= 2\frac{\zeta_{\sigma(j)}(\zeta_{\sigma(j)}-\zeta_{\sigma(k)})\zeta_{\sigma(k)}}{|\zeta_{\sigma(j)}||\zeta_{\sigma(k)}|}\sqrt{\frac{\omega_{\sigma(j)}\omega_{\sigma(k)}}{(1+\zeta^2_{\sigma(j)})(1+\zeta^2_{\sigma(k)})}}.
\end{aligned} \tag{2.1.6}$$

Plugging these expressions into Eq. (2.1.5), we find the following structure for the MHV amplitude

$$A_n\Big(1^-,2^-,\sigma(3^+,\dots,n^+)\Big) = is\left(\prod_{i=3}^{n}\frac{1+\zeta_i^2}{\omega_i}\right) f_\sigma(\zeta_3,\dots,\zeta_n), \tag{2.1.7}$$

where $f_\sigma(\zeta_3, \ldots, \zeta_n)$ is a rational homogeneous function of degree $2-n$:

$$f_\sigma(\lambda\zeta_3, \ldots, \lambda\zeta_n) = \lambda^{2-n} f_\sigma(\zeta_3, \ldots, \zeta_n), \tag{2.1.8}$$

for any $\lambda \neq 0$. Thus, the color-dressed planar tree-level n-gluon amplitude reads

$$A_n = is(ig)^{n-2} \left(\prod_{i=3}^{n} \frac{1+\zeta_i^2}{\omega_i} \right) \sum_{\sigma \in S_{n-2}} c_\sigma f_\sigma(\zeta_3, \ldots, \zeta_n), \tag{2.1.9}$$

and the planar zeros are determined by the homogeneous equation

$$\sum_{\sigma \in S_{n-2}} c_\sigma f_\sigma(\zeta_3, \ldots, \zeta_n) = 0. \tag{2.1.10}$$

Reducing denominators in this equation, the condition for the existence of planar zeros is recast into a homogeneous polynomial of degree $\frac{1}{2}(n-2)(n-3)$ in the stereographic variables.

In the case of graviton scattering, the analysis carried out in Chapter 1 showed that the planar, five-graviton amplitude automatically vanishes without imposing any further condition on the kinematics of the outgoing particles.

The aim of this chapter is to further investigate these issues, focusing on the conditions under which the projective structure of the planar zeros is preserved. We will see that the equation determining them is invariant under a simultaneous rescaling of the outgoing stereographic coordinates for theories with gauge invariance, even when matter scalar fields are introduced. The resulting projective curves are of the same type as the ones found for gluon scattering in Chapter 1. Pure scalar theories, on the other hand, give rise to equations for the existence of planar zeros which are not homogeneous in the stereographic coordinates.

In the case of graviton scattering, we trace the vanishing of the planar five-graviton amplitude found in Ch. 1 to the fact that the amplitude becomes effectively three-dimensional in this limit. According to a general result of [39], odd-multiplicity three-dimensional gravitation amplitudes vanish due to helicity non-conservation. This conclusion is confirmed by a computation of the planar limit of the helicity-preserving six-graviton amplitude, which gives a nonzero result. We revisit the amplitudes for the scattering of scalars with graviton emission to find that they also vanish identically in the planar limit, similarly to what happens with the five-graviton amplitude computed in Ch. 1.

We also study the corrections to planar zeros associated with the ultraviolet completions of gauge theories and gravity provided by string theory. For Yang-Mills scattering, we compute the five gauge bosons disk amplitude using the methods developed in Ref. [40,41]. Expanding the generalized Euler integrals in powers of the inverse string tension, we find that the planar zero condition found in the field theory limit gets corrected by equations which fail to preserve the projective structure found in Ch. 1. In the context of gravity amplitudes, we compute the five-graviton closed string amplitude on the sphere and its α' expansion using the single-valued projection [42–44]. Unlike its field theory limit, the planar graviton string amplitude is generically nonzero, thus avoiding the consequences of the theorem proved in [39].

The plan of the chapter is as follows. In Section 2.2 we analyze planar zeros in a cubic non-gauge massless scalar theory transforming as bi-adjoints under a global symmetry group. We particularize our analysis to the case of a ϕ^3 theory, where we study the structure of the curves determining the planar zeros. We dedicate Section 2.3 to the study of planar zeros in a theory of scalars coupled to a gauge field. Having completed our presentation of the properties of planar zeros in gauge field theories, we proceed in Section 2.4 to the computation of the α' corrections to the five-gluon amplitude in the planar limit.

In Section 2.5 we turn our attention to the planar zeros of gravitational scattering amplitudes, considering the collision of two scalars, both distinguishable and indistinguishable, with emission of a graviton. Here we also analyze the helicity-preserving six-graviton amplitude, which turns out to be nonzero in the planar limit. Section 2.6 is devoted to the study of the α' corrections to the planar five graviton tree level amplitude. Finally, in Section 2.7 we discuss how the projective properties of planar zeros in theories with gauge invariance emerge from the structure of the amplitude in the soft limit. Our conclusions are summarized in Section 2.8. To avoid cluttering the main text with cumbersome expressions, some long equations have been deferred to the Appendix.

2.2 Pure scalar theories

We begin with the analysis of a pure scalar theory transforming in the biadjoint representation of a generic global symmetry group $G \times \overline{G}$, with action

$$S = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu \Phi^{a\overline{a}} \partial^\mu \Phi^{a\overline{a}} - \frac{\lambda}{3!} f^{abc} \overline{f}^{\overline{a}\overline{b}\overline{c}} \Phi^{a\overline{a}} \Phi^{b\overline{b}} \Phi^{c\overline{c}} \right), \tag{2.2.1}$$

and study the tree-level, five point amplitude. A very economic way of obtaining this amplitude is by using the so-called zeroth-copy prescription [45], consisting in replacing kinematic numerators in the pure-gauge theory amplitude with a second copy of the color factors

$$\mathcal{A}_{n,\text{gauge}} = g_{\text{YM}}^{n-2} \sum_{i\in\Gamma} \frac{c_i n_i}{\prod_\alpha s_{i,\alpha}} \qquad \Longrightarrow \qquad \mathcal{A}_{n,\text{scalar}} = \lambda^{n-2} \sum_{i\in\Gamma} \frac{c_i \overline{c}_i}{\prod_\alpha s_{i,\alpha}}, \tag{2.2.2}$$

where c_i, $\overline{c}_i$ are the color factors of G and $\overline{G}$ respectively. With this, we find

$$\begin{aligned} \mathcal{A}_{5,\text{scalars}} = i\lambda^3 \Bigg(& \frac{c_1\overline{c}_1}{s_{12}s_{45}} + \frac{c_2\overline{c}_2}{s_{23}s_{15}} + \frac{c_3\overline{c}_3}{s_{34}s_{12}} + \frac{c_4\overline{c}_4}{s_{45}s_{23}} + \frac{c_5\overline{c}_5}{s_{15}s_{34}} + \frac{c_6\overline{c}_6}{s_{14}s_{25}} + \frac{c_7\overline{c}_7}{s_{13}s_{25}} \\ & + \frac{c_8\overline{c}_8}{s_{24}s_{13}} + \frac{c_9\overline{c}_9}{s_{35}s_{24}} + \frac{c_{10}\overline{c}_{10}}{s_{14}s_{35}} + \frac{c_{11}\overline{c}_{11}}{s_{15}s_{24}} + \frac{c_{12}\overline{c}_{12}}{s_{12}s_{35}} + \frac{c_{13}\overline{c}_{13}}{s_{23}s_{14}} + \frac{c_{14}\overline{c}_{14}}{s_{25}s_{34}} + \frac{c_{15}\overline{c}_{15}}{s_{13}s_{45}} \Bigg), \end{aligned} \tag{2.2.3}$$

where we have defined the kinematic invariants

$$s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j \quad \text{where} \quad i < j, \tag{2.2.4}$$

and the color factors are given by

$$\begin{aligned}
c_1 &= f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5}\,, & c_2 &= f^{a_1a_5b}f^{ba_4c}f^{ca_3a_2}\,, & c_3 &= f^{a_3a_4b}f^{ba_5c}f^{ca_1a_2}\,,\\
c_4 &= f^{a_4a_5b}f^{ba_1c}f^{ca_2a_3}\,, & c_5 &= f^{a_5a_1b}f^{ba_2c}f^{ca_3a_4}\,, & c_6 &= f^{a_1a_4b}f^{ba_3c}f^{ca_5a_2}\,,\\
c_7 &= f^{a_1a_3b}f^{ba_4c}f^{ca_5a_2}\,, & c_8 &= f^{a_1a_3b}f^{ba_5c}f^{ca_4a_2}\,, & c_9 &= f^{a_3a_5b}f^{ba_1c}f^{ca_2a_4}\,,\\
c_{10} &= f^{a_4a_1b}f^{ba_2c}f^{ca_3a_5}\,, & c_{11} &= f^{a_1a_5b}f^{ba_3c}f^{ca_4a_2}\,, & c_{12} &= f^{a_3a_5b}f^{ba_4c}f^{ca_1a_2}\,,\\
c_{13} &= f^{a_1a_4b}f^{ba_5c}f^{ca_3a_2}\,, & c_{14} &= f^{a_5a_2b}f^{ba_1c}f^{ca_3a_4}\,, & c_{15} &= f^{a_1a_3b}f^{ba_2c}f^{ca_4a_5}\,.
\end{aligned} \tag{2.2.5}$$

As in Ch. 1, we work in a center-of-mass reference frame in which the incoming particles have momenta given by (2.1.3), whereas the momenta of the three outgoing particles are parametrized using the stereographic coordinates as given in Eq. (2.1.4). In our convention all momenta enter the diagram. We consider planar scattering processes taking place on the plane $y = 0$, i.e. $\zeta_a = \overline{\zeta}_a$ for $a = 3, 4, 5$. In the case of the five-point amplitude, imposing energy-momentum conservation completely determines the energies of the outgoing particles:

$$\begin{aligned}
\omega_3 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_3^2)(1+\zeta_4\zeta_5)}{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)}\,,\\
\omega_4 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_4^2)(1+\zeta_3\zeta_5)}{(\zeta_4-\zeta_3)(\zeta_4-\zeta_5)}\,,\\
\omega_5 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_5^2)(1+\zeta_3\zeta_4)}{(\zeta_5-\zeta_3)(\zeta_5-\zeta_4)}\,.
\end{aligned} \tag{2.2.6}$$

Writing the kinematic invariants in (2.2.3) using our parametrization of the momenta, we arrive at the following equation for the planar five-point scalar amplitude

$$\mathcal{A}_{5,\text{scalar}}\Big|_{\text{planar}} = \left(\frac{i\lambda^3}{s^2}\right)\frac{P_{10}(\zeta_3,\zeta_4,\zeta_5)}{\zeta_3^2\zeta_4^2\zeta_5^2(1+\zeta_3\zeta_4)(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)}\,, \tag{2.2.7}$$

where $P_{10}(\zeta_3, \zeta_4, \zeta_5)$ is a degree-ten polynomial whose coefficients depend on the color factors. The explicit expression for this polynomial can be found in Eq. (B.1). The amplitude has collinear singularities at $\zeta_a \to 0, \infty$ together with soft poles at $\zeta_a\zeta_b \to -1$ (with $a < b$) where the energy of one of the outgoing particles tend to zero. In addition, the collinear limits $\zeta_a \to \zeta_b$ lead to a divergence of the energies of the outgoing particles.

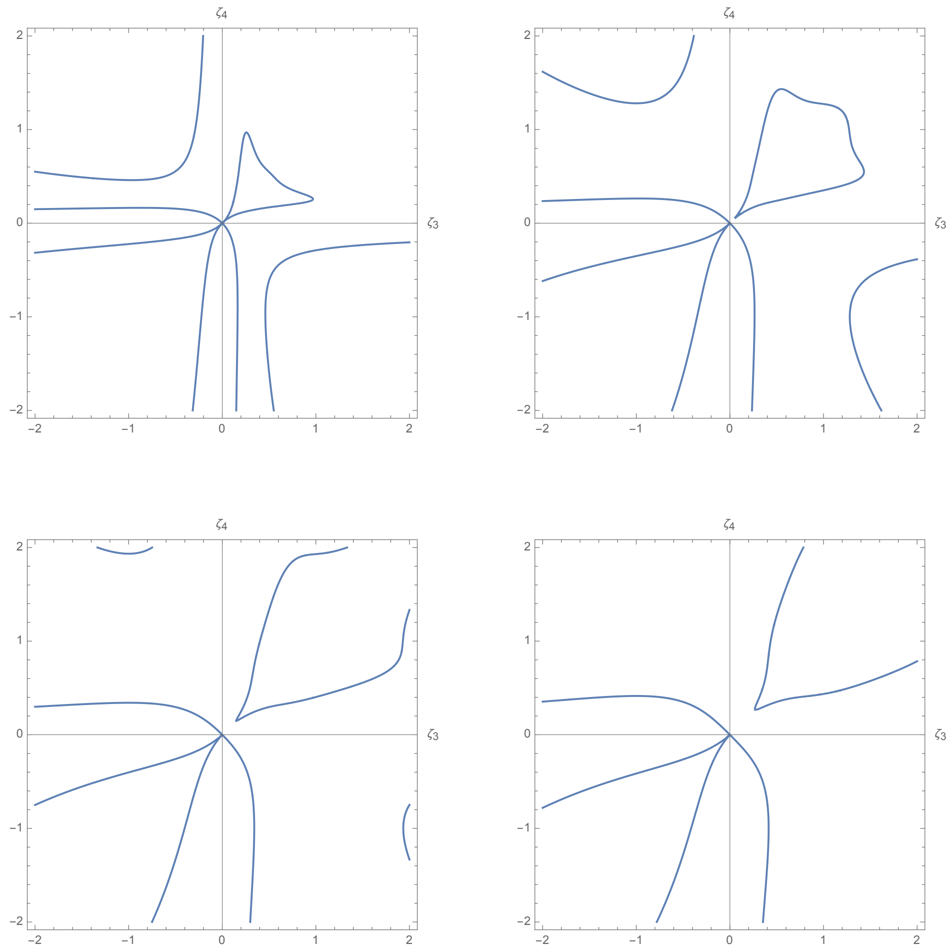


Figure 2.1: Sections of the loci of planar zeros $P_{10}^{\phi^3}(\zeta_3, \zeta_4, \zeta_5) = 0$ for $\zeta_5 = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

Planar zeros are thus determined by the equation

$$P_{10}(\zeta_3, \zeta_4, \zeta_5) = 0\ . \tag{2.2.8}$$

Inspecting Eq. (B.1.1), we find that, unlike the case of pure gauge theories studied in the previous chapter, this equation is not homogeneous in the stereographic coordinates, since it contains monomials of both degree 10 and 8. Thus, unlike the case of pure gauge theories studied in Ch. 1, planar zeros are no longer determined by a projective curve[1].

To study the corresponding geometric loci of planar zeros, we focus on the five-point amplitude for ϕ^3 theory, which can be retrieved from Eq. (2.2.3) by setting all color factors equal to one, $c_i\bar{c}_i \to 1$. In this case, the expression for $P_{10}(\zeta_3, \zeta_4, \zeta_5)$ in (2.2.7) somewhat

[1]Still, looking at (B.1.1), we see that the equation $P_{10}(\zeta_3, \zeta_4, \zeta_5) = 0$ is homogeneous in the color factors. As a consequence, after a proper normalization of the group theory generators, planar zeros are determined by an equation with integer coefficients.

simplifies to

$$\begin{aligned}
P_{10}^{\phi^3}(\zeta_3,\zeta_4,\zeta_5) = {} & \zeta_3^6\zeta_4^3\zeta_5 - \zeta_3^6\zeta_4^2\zeta_5^2 + \zeta_3^6\zeta_4\zeta_5^3 - \zeta_3^5\zeta_4^4\zeta_5 - \zeta_3^5\zeta_4^3\zeta_5^2 + \zeta_3^5\zeta_4^3 \\
& - \zeta_3^5\zeta_4^2\zeta_5^3 - \zeta_3^5\zeta_4^2\zeta_5 - \zeta_3^5\zeta_4\zeta_5^4 - \zeta_3^5\zeta_4\zeta_5^2 + \zeta_3^5\zeta_5^3 - \zeta_3^4\zeta_4^5\zeta_5 \\
& + 4\zeta_3^4\zeta_4^4\zeta_5^2 - \zeta_3^4\zeta_4^4 - \zeta_3^4\zeta_4^3\zeta_5^3 - \zeta_3^4\zeta_4^3\zeta_5 + 4\zeta_3^4\zeta_4^2\zeta_5^4 + 4\zeta_3^4\zeta_4^2\zeta_5^2 \\
& - \zeta_3^4\zeta_4\zeta_5^5 - \zeta_3^4\zeta_4\zeta_5^3 - \zeta_3^4\zeta_5^4 + \zeta_3^3\zeta_4^6\zeta_5 - \zeta_3^3\zeta_4^5\zeta_5^2 + \zeta_3^3\zeta_4^5 \\
& - \zeta_3^3\zeta_4^4\zeta_5^3 - \zeta_3^3\zeta_4^4\zeta_5 - \zeta_3^3\zeta_4^3\zeta_5^4 - \zeta_3^3\zeta_4^3\zeta_5^2 - \zeta_3^3\zeta_4^2\zeta_5^5 - \zeta_3^3\zeta_4^2\zeta_5^3 \\
& + \zeta_3^3\zeta_4\zeta_5^6 - \zeta_3^3\zeta_4\zeta_5^4 + \zeta_3^3\zeta_5^5 - \zeta_3^2\zeta_4^6\zeta_5^2 - \zeta_3^2\zeta_4^5\zeta_5^3 - \zeta_3^2\zeta_4^5\zeta_5 \\
& + 4\zeta_3^2\zeta_4^4\zeta_5^4 + 4\zeta_3^2\zeta_4^4\zeta_5^2 - \zeta_3^2\zeta_4^3\zeta_5^5 - \zeta_3^2\zeta_4^3\zeta_5^3 - \zeta_3^2\zeta_4^2\zeta_5^6 \\
& + 4\zeta_3^2\zeta_4^2\zeta_5^4 - \zeta_3^2\zeta_4\zeta_5^5 + \zeta_3\zeta_4^6\zeta_5^3 - \zeta_3\zeta_4^5\zeta_5^4 - \zeta_3\zeta_4^5\zeta_5^2 - \zeta_3\zeta_4^4\zeta_5^5 \\
& - \zeta_3\zeta_4^4\zeta_5^3 + \zeta_3\zeta_4^3\zeta_5^6 - \zeta_3\zeta_4^3\zeta_5^4 - \zeta_3\zeta_4^2\zeta_5^5 + \zeta_4^5\zeta_5^3 - \zeta_4^4\zeta_5^4 + \zeta_4^3\zeta_5^5\,.
\end{aligned} \tag{2.2.9}$$

Notice that the polynomial is symmetric in all its entries, as expected by Bose symmetry. This implies that when studying the curves, we can consider sections of constant ζ_5 without loss of generality. In Fig. 2.1 we have plotted the sections for $\zeta_5 = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1. The curves are more complicated than in the pure gauge case and include singular points.

2.3 Planar zeros in scalar QCD

The results of the previous section show that the projective character of planar zeros in the five-point function gluon amplitude disappears when considering pure scalar theories, even in the presence of global symmetries. This is indeed due to the absence of derivative couplings, which render the numerators appearing in the scalar amplitude (2.2.3) trivial. It is therefore tempting to conclude that, despite the similarities in the topologies contributing to both the gauge and scalar amplitudes, the projective nature of the equation determining the planar zeros is a consequence of gauge invariance.

2.3.1 Distinguishable scalars

To further explore this possibility, we study now the presence of planar zeros in scalar QCD (sQCD), in particular in the scattering of two distinct scalars with emission of a gluon in the final state. This process has been studied in Ref. [46]. We label momenta and color quantum numbers according to

$$\Phi(p_1, j) + \Phi'(p_2, n) \longrightarrow \Phi(p_3, i) + \Phi'(p_4, m) + g(p_5, a, \epsilon_+), \tag{2.3.1}$$

where $\epsilon_+(p_5)$ indicates the polarization vector of the gluon. We slightly modify the conventions of Ref. [46], and consider all momenta as incoming. The amplitude takes the form

$$\mathcal{A} = g^3 \left(\frac{C_1 n_1}{s_{24}s_{35}} + \frac{C_2 n_2}{s_{24}s_{15}} + \frac{C_3 n_3}{s_{24}} + \frac{C_4 n_4}{s_{13}s_{45}} + \frac{C_5 n_5}{s_{13}s_{25}} + \frac{C_6 n_6}{s_{13}} + \frac{C_7 n_7}{s_{13}s_{24}} \right), \tag{2.3.2}$$

where now there are seven different color factors

$$\begin{aligned}
&C_1 = T^a_{ik}T^b_{kj}\overline{T}^b_{mn}, \qquad C_2 = T^b_{ik}T^a_{kj}\overline{T}^b_{mn}, \qquad C_3 = T^a_{ik}T^b_{kj}\overline{T}^b_{mn} + T^b_{ik}T^a_{kj}\overline{T}^b_{mn}, \\
&C_4 = T^b_{ij}\overline{T}^a_{mk}\overline{T}^b_{kn}, \qquad C_5 = T^b_{ij}\overline{T}^b_{mk}\overline{T}^a_{kn}, \qquad C_6 = T^b_{ij}\overline{T}^a_{mk}\overline{T}^b_{kn} + T^b_{ij}\overline{T}^b_{mk}\overline{T}^a_{kn}, \\
&C_7 = if^{abc}T^b_{ij}\overline{T}^c_{mn}.
\end{aligned} \tag{2.3.3}$$

We allow for the possibility of the two scalars transforming in different representations of the gauge group. The color factors satisfy four Jacobi identities

$$\begin{aligned}
C_1 - C_2 + C_7 = 0, \qquad C_1 + C_2 - C_3 = 0, \\
C_4 - C_5 - C_7 = 0, \qquad C_4 + C_5 - C_6 = 0.
\end{aligned} \tag{2.3.4}$$

These relations can be used to express the five-point amplitudes in terms of only three independent color factors, that we take C_1, C_2, and C_4. Namely,

$$\begin{aligned}
C_3 = C_1 + C_2, \qquad C_5 = C_1 - C_2 + C_4, \\
C_6 = C_1 - C_2 + 2C_4, \qquad C_7 = -C_1 + C_2.
\end{aligned} \tag{2.3.5}$$

Again, we work in the center-of-mass reference frame and use (2.1.4) to express outgoing momenta in terms of the stereographic coordinates. In the planar limit $\zeta_a = \overline{\zeta}_a$ we take the gluon polarization vector to be

$$\epsilon_\pm = \pm\frac{1}{\sqrt{2}} \left(0, \frac{\zeta_5^2 - 1}{1 + \zeta_5^2}, \mp i, -\frac{2\zeta_5}{1+\zeta_5^2} \right), \tag{2.3.6}$$

which indeed satisfies $p_5 \cdot \epsilon_\pm(p_5) = 0$. In the following, we specialize our analysis to a positive helicity gluon, $\epsilon \equiv \epsilon_+$. With this, the numerators in the amplitude (2.3.2) take the following form in the planar limit

$$\begin{aligned}
n_1 &= -2i[(p_1 - p_3 - p_5)\cdot(p_2 - p_4)][p_3 \cdot \epsilon_+(p_5)] \\
&= \frac{i\sqrt{2}s^{\frac{3}{2}}(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)}{(1+\zeta_5^2)(\zeta_3-\zeta_4)^2(\zeta_4-\zeta_5)}(-1+\zeta_3\zeta_4-\zeta_4^2-2\zeta_3\zeta_5+\zeta_4\zeta_5), \\
n_2 &= 2i[(p_1 - p_3 + p_5)\cdot(p_2 - p_4)][p_1 \cdot \epsilon_+(p_5)]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{i\sqrt{2}s^{\frac{3}{2}}\zeta_5(1+\zeta_4\zeta_5)}{(1+\zeta_5^2)(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)}(-\zeta_3+\zeta_4+\zeta_3^2\zeta_4-2\zeta_3\zeta_5+\zeta_3\zeta_4\zeta_5),\\
n_3 &= -i(p_2-p_4)\cdot\epsilon_+(p_5)\\
&= \frac{i\sqrt{s}}{\sqrt{2}(1+\zeta_5^2)(\zeta_3-\zeta_4)}(1+2\zeta_3\zeta_5+\zeta_3\zeta_4\zeta_5^2),\\
n_4 &= -2i[(p_2-p_4-p_5)\cdot(p_1-p_3)][p_4\cdot\epsilon_+(p_5)] \qquad (2.3.7)\\
&= -\frac{i\sqrt{2}s^{\frac{3}{2}}(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)}{(1+\zeta_5^2)(\zeta_3-\zeta_4)^2(\zeta_3-\zeta_5)}(2\zeta_3^2-\zeta_3\zeta_4-\zeta_3\zeta_5+\zeta_4\zeta_5+\zeta_3^2\zeta_4\zeta_5),\\
n_5 &= 2i[(p_2-p_4+p_5)\cdot(p_1-p_3)][p_2\cdot\epsilon_+(p_5)]\\
&= -\frac{i\sqrt{2}s^{\frac{3}{2}}\zeta_5(1+\zeta_3\zeta_5)}{(1+\zeta_5^2)(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)}(-2\zeta_3+\zeta_4+\zeta_5-\zeta_3\zeta_4\zeta_5+\zeta_4^2\zeta_5),\\
n_6 &= -i(p_1-p_3)\cdot\epsilon_+(p_5)\\
&= -\frac{i\sqrt{s}}{\sqrt{2}(1+\zeta_5^2)(\zeta_3-\zeta_4)}(1+2\zeta_3\zeta_5+\zeta_3\zeta_4\zeta_5^2),\\
n_7 &= -i\Big\{[(p_2-p_4)\cdot(p_1+p_3-p_5)][(p_1-p_3)\cdot\epsilon_+(p_5)]\\
&- \ [(p_1-p_3)\cdot(p_2-p_4)][(p_3+p_1-p_4-p_2)\cdot\epsilon_+(p_5)]\\
&- \ [(p_1-p_3)\cdot(p_4+p_2-p_5)][(p_2-p_4)\cdot\epsilon_+(p_5)]\Big\}\\
&= \frac{is^{\frac{3}{2}}}{\sqrt{2}(1+\zeta_5^2)(\zeta_3-\zeta_4)^2(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)}\Big(-\zeta_3^3\zeta_4^3\zeta_5^3-4\zeta_3^3\zeta_4^2\zeta_5^2-4\zeta_3^3\zeta_4\zeta_5\\
&+\zeta_3^2\zeta_4^3\zeta_5^4+\zeta_3^2\zeta_4^3\zeta_5^2+8\zeta_3^2\zeta_4^2\zeta_5^3-\zeta_3^2\zeta_4^2\zeta_5-4\zeta_3^2\zeta_4\zeta_5^4+8\zeta_3^2\zeta_4\zeta_5^2-4\zeta_3^2\zeta_4\\
&-4\zeta_3^2\zeta_5^3-4\zeta_3\zeta_4^3\zeta_5^3+2\zeta_3\zeta_4^3\zeta_5+3\zeta_3\zeta_4^2\zeta_5^4+2\zeta_3\zeta_4^2\zeta_5^2+3\zeta_3\zeta_4^2-\zeta_3\zeta_4\zeta_5^3\\
&+8\zeta_3\zeta_4\zeta_5-4\zeta_3\zeta_5^2-4\zeta_4^3\zeta_5^2+2\zeta_4^2\zeta_5^3-4\zeta_4^2\zeta_5+\zeta_4\zeta_5^2+\zeta_4-\zeta_5\Big).
\end{aligned}
$$

As a nontrivial test of the previous equations, it can be checked that the amplitude satisfies the gauge Ward identity. Combining the numerators with the expressions for the kinematic invariants we arrive at the following form of the tree-level amplitude of distinct scalars with gluon emission in the limit of planar scattering:

$$
\begin{aligned}
\mathcal{A}_5\Big|_{\text{planar}} &= \frac{i\sqrt{2}g^3(2\zeta_3-\zeta_4)}{\sqrt{s}\zeta_4\zeta_5(1+\zeta_3\zeta_4)}\\
&\times\Big[(C_1-C_2+C_4)\zeta_3\zeta_4-(C_1-C_2)\zeta_3\zeta_5+(C_2-C_4)\zeta_4\zeta_5-C_2\zeta_5^2\Big]. \qquad (2.3.8)
\end{aligned}
$$

Similarly to what happened for pure gauge theories in Ch. 1, planar zeros are determined by a homogeneous cubic polynomial

$$(2\zeta_3 - \zeta_4)\Big[(C_1 - C_2 + C_4)\zeta_3\zeta_4 - (C_1 - C_2)\zeta_3\zeta_5 + (C_2 - C_4)\zeta_4\zeta_5 - C_2\zeta_5^2\Big] = 0. \quad (2.3.9)$$

An important difference with the pure gauge theory case is, however, that the polynomial factorizes. One of the factors, the trivial branch, is linear and independent of the color factors of the interacting particles,

$$2\zeta_3 - \zeta_4 = 0. \quad (2.3.10)$$

It is also independent of the direction of flight of the emitted gluon. The second, non-trivial branch is a quadratic equation

$$(C_1 - C_2 + C_4)\zeta_3\zeta_4 - (C_1 - C_2)\zeta_3\zeta_5 + (C_2 - C_4)\zeta_4\zeta_5 - C_2\zeta_5^2 = 0, \quad (2.3.11)$$

whose coefficients depend on the three independent color factors.

Being also homogeneous in the color factors, the polynomial (2.3.9) defines an integer curve in the projective plane defined by the coordinates $(\zeta_3, \zeta_4, \zeta_5)$. It seems natural now to single out the direction of flight of the emitted gluon and study this curve in the patch centered around the point $(0, 0, 1)$ using the coordinates

$$(\zeta_3, \zeta_4, \zeta_5) = \lambda(U, V, 1). \quad (2.3.12)$$

Now, the trivial branch of planar zeros is determined by the straight line

$$2U - V = 0, \quad (2.3.13)$$

whereas the non-trivial quadratic curve takes the form

$$(C_1 - C_2 + C_4)UV - (C_1 - C_2)U + (C_2 - C_4)V - C_2 = 0. \quad (2.3.14)$$

The quadratic curve (2.3.14) can be easily classified for a generic gauge group in terms of the three invariants (Δ, δ, I) and the semiinvariant σ (see, for example, [47]) defined by

$$\begin{aligned}
\Delta &= \frac{1}{4}C_1C_4(C_1 - C_2 + C_4), \\
\delta &= -\frac{1}{4}(C_1 - C_2 + C_4)^2, \\
I &= 0, \\
\sigma &= -\frac{1}{4}(C_1 - C_2)^2 - \frac{1}{4}(C_2 - C_4)^2.
\end{aligned} \quad (2.3.15)$$

Since $\delta \leq 0$, no ellipses are possible. It is also impossible to have $\delta = 0$ with $\Delta \neq 0$, so parabolas are ruled out as well. Thus, the only possible class of curves are hyperbolas ($\Delta \neq 0$, $\delta < 0$), intersecting lines ($\Delta = 0$, $\delta < 0$), or parallel lines ($\Delta = \delta = 0$, $\sigma < 0$). Notice that this classification is valid for *all* gauge groups and *all* representations of the scalar fields.

As an illustrative example, we study the case of two scalars with charges e and e' coupled to a photon. This correspond to having the $U(1)$ generators

$$T^1_{ij} = e\delta_{ij}, \qquad \overline{T}^1_{mn} = e'\delta_{mn}, \tag{2.3.16}$$

giving the following values for the color factors

$$C_1 = C_2 = \frac{1}{2}C_3 = e^2 e', \qquad C_4 = C_5 = \frac{1}{2}C_6 = ee'^2, \qquad C_7 = 0. \tag{2.3.17}$$

In the patch centered around $U = V = 0$, the projective curve determining the planar zeros is given by

$$ee'^2 UV + ee'(e - e')V - e^2 e' = 0. \tag{2.3.18}$$

Since

$$\Delta = \frac{1}{4}e^4 e'^5 \neq 0, \qquad \delta = -\frac{1}{4}e^2 e'^4 < 0, \tag{2.3.19}$$

the loci of planar zeros are hyperbolas with asymptotes along the coordinates axes and whose center is located at the point

$$(U_0, V_0) = \left(\frac{e' - e}{e'}, 0\right). \tag{2.3.20}$$

A particularly simple case arises when we consider that both scalars, though distinct, have the same electric charge, $e = e'$. In this case the curve is given by $UV = 1$.

2.3.2 Indistinguishable scalars

The previous analysis of the scattering amplitude of distinct scalars coupled to a gauge field in an arbitrary representation illustrates how the derivative couplings required by gauge invariance are enough to restore the projective nature of planar zeros, that was absent in the pure scalar theories studied in Section 2.2. This is also the case when considering sQCD with a single scalar field in the adjoint representation of the gauge group. We consider again a five-point amplitude corresponding to the process [48]

$$\Phi(p_1, a_1) + \Phi(p_2, a_2) \longrightarrow \Phi(p_3, a_3) + \Phi(p_4, a_4) + g(p_5, a_5, \epsilon). \tag{2.3.21}$$

After all quartic couplings are resolved in terms of trivalent vertices, the 15 topologies contributing to this amplitude are the ones already encountered in both pure Yang-Mills theories and the scalar theories studied in Section 2.2. The amplitude takes the form

$$\mathcal{A}_5 = g^3\left(\frac{c_1n_1}{s_{12}s_{45}}+\frac{c_2n_2}{s_{23}s_{15}}+\frac{c_3n_3}{s_{34}s_{12}}+\frac{c_4n_4}{s_{45}s_{23}}+\frac{c_5n_5}{s_{15}s_{34}}+\frac{c_6n_6}{s_{14}s_{25}}+\frac{c_7n_7}{s_{13}s_{25}}+\frac{c_8n_8}{s_{24}s_{13}}\right.$$
$$\left.+\frac{c_9n_9}{s_{35}s_{24}}+\frac{c_{10}n_{10}}{s_{14}s_{35}}+\frac{c_{11}n_{11}}{s_{15}s_{24}}+\frac{c_{12}n_{12}}{s_{12}s_{35}}+\frac{c_{13}n_{13}}{s_{23}s_{14}}+\frac{c_{14}n_{14}}{s_{25}s_{34}}+\frac{c_{15}n_{15}}{s_{13}s_{45}}\right), \qquad (2.3.22)$$

where the color factors are the ones defined in (2.2.5), while the numerators are given by

$$\begin{aligned}
n_1 &= (p_4+p_5)^2[(p_2-p_1)\cdot\epsilon_+(p_5)] + 2(p_2-p_1)\cdot(p_3-p_4-p_5)[p_4\cdot\epsilon_+(p_5)],\\
n_2 &= -(p_1+p_5)^2[(p_2-p_3)\cdot\epsilon_+(p_5)] - 2(p_2-p_3)\cdot(-p_1+p_4-p_5)[p_1\cdot\epsilon_+(p_5)],\\
n_3 &= -(p_2-p_1)\cdot(p_3-p_4)[(p_1+p_2-p_3-p_4)\cdot\epsilon_+(p_5)]\\
&- \ (p_3-p_4)\cdot(-p_1-p_2+p_5)[(p_2-p_1)\cdot\epsilon_+(p_5)]\\
&- \ (p_2-p_1)\cdot(p_3+p_4-p_5)[(p_3-p_4)\cdot\epsilon_+(p_5)],\\
n_4 &= (p_4+p_5)^2[(p_2-p_3)\cdot\epsilon_+(p_5)] + 2(p_2-p_3)\cdot(p_1-p_4-p_5)[p_4\cdot\epsilon_+(p_5)],\\
n_5 &= -(p_1+p_5)^2[(p_3-p_4)\cdot\epsilon_+(p_5)] - 2(p_3-p_4)\cdot(-p_1+p_2-p_5)[p_1\cdot\epsilon_+(p_5)],\\
n_6 &= -(p_2+p_5)^2[(p_4-p_1)\cdot\epsilon_+(p_5)] - 2(p_4-p_1)\cdot(-p_2+p_3-p_5)[p_2\cdot\epsilon_+(p_5)],\\
n_7 &= -(p_2+p_5)^2[(p_3-p_1)\cdot\epsilon_+(p_5)] - 2(p_3-p_1)\cdot(-p_2+p_4-p_5)[p_2\cdot\epsilon_+(p_5)],\\
n_8 &= (p_2-p_4)\cdot(p_1+p_3-p_5)[(p_3-p_1)\cdot\epsilon_+(p_5)]\\
&+ \ (p_3-p_1)\cdot(-p_2-p_4+p_5)[(p_2-p_4)\cdot\epsilon_+(p_5)]\\
&+ \ (p_3-p_1)\cdot(p_2-p_4)[(-p_1+p_2-p_3+p_4)\cdot\epsilon_+(p_5)],\\
n_9 &= -(p_3+p_5)^2[(p_4-p_2)\cdot\epsilon_+(p_5)] - 2(p_4-p_2)\cdot(-p_3+p_1-p_5)[p_3\cdot\epsilon_+(p_5)],\\
n_{10} &= -(p_3+p_5)^2[(p_4-p_1)\cdot\epsilon_+(p_5)] - 2(p_4-p_1)\cdot(p_2-p_3-p_5)[p_3\cdot\epsilon_+(p_5)],\\
n_{11} &= -(p_1+p_5)^2[(p_2-p_4)\cdot\epsilon_+(p_5)] - 2(p_2-p_4)\cdot(-p_1+p_3-p_5)[p_1\cdot\epsilon_+(p_5)],\\
n_{12} &= -(p_3+p_5)^2[(p_2-p_1)\cdot\epsilon_+(p_5)] - 2(p_2-p_1)\cdot(-p_3+p_4-p_5)[p_3\cdot\epsilon_+(p_5)],\\
n_{13} &= (p_2-p_3)\cdot(p_4-p_1)[(-p_1+p_2+p_3-p_4)\cdot\epsilon(p_5)]\\
&+ \ (p_4-p_1)\cdot(-p_2-p_3+p_5)[(p_2-p_3)\cdot\epsilon_+(p_5)]\\
&+ \ (p_2-p_3)\cdot(p_1+p_4-p_5)[(p_4-p_1)\cdot\epsilon_+(p_5)],\\
n_{14} &= -(p_2+p_5)^2[(p_3-p_4)\cdot\epsilon_+(p_5)] - 2(p_3-p_4)\cdot(-p_2+p_1-p_5)[p_2\cdot\epsilon_+(p_5)],\\
n_{15} &= (p_4+p_5)^2[(p_3-p_1)\cdot\epsilon_+(p_5)] + 2(p_3-p_1)\cdot(p_2-p_4-p_5)[p_4\cdot\epsilon_+(p_5)].
\end{aligned} \qquad (2.3.23)$$

We have assumed again that the emitted gluon has positive helicity.

A long but straightforward evaluation of the amplitude in the planar limit gives the result

$$\mathcal{A}_5\bigg|_{\text{planar}} = -\frac{2\sqrt{2}g^3}{\sqrt{s}}\frac{(\zeta_3^2-\zeta_3\zeta_4+\zeta_4^2)}{(\zeta_3-\zeta_4)(1+\zeta_3\zeta_4)}\left[-c_2\frac{\zeta_5-\zeta_3}{\zeta_3} - c_6\frac{\zeta_4-\zeta_5}{\zeta_5} + c_7\frac{\zeta_3-\zeta_5}{\zeta_5}\right.$$

$$- c_8 \frac{\zeta_3 - \zeta_4}{\zeta_4} + c_{11} \frac{\zeta_5 - \zeta_4}{\zeta_4} + c_{13} \frac{\zeta_4 - \zeta_3}{\zeta_3} \Bigg] . \tag{2.3.24}$$

The prefactor $\zeta_3^2 - \zeta_3\zeta_4 + \zeta_4^2$ does not have real nontrivial zeros, corresponding to two complex straight lines in the (ζ_3, ζ_4) plane. After multiplying by $\zeta_3\zeta_4\zeta_5$, which does not introduce any spurious physical zeros, we arrive at the cubic homogeneus equation

$$\begin{aligned} &c_7\zeta_3^2\zeta_4 - c_8\zeta_3^2\zeta_5 - c_6\zeta_3\zeta_4^2 + c_{11}\zeta_3\zeta_5^2 \\ &\qquad + (c_2 + c_6 - c_7 + c_8 - c_{11} - c_{13})\zeta_3\zeta_4\zeta_5 + c_{13}\zeta_4^2\zeta_5 - c_2\zeta_4\zeta_5^2 = 0. \end{aligned} \tag{2.3.25}$$

Interestingly, the condition (2.3.25) for the existence of planar zeros in the scattering of two indistinguishable scalars with the emission of a gluon is identical to the one found for the five-gluon scattering amplitude in Eq. (1.3.6). Therefore, the same analysis of the curves in Ch. 1 for various gauge groups is valid here.

2.4 String corrections to gauge theory planar zeros

It would be interesting to see how the planar zeros of (super) Yang-Mills theories get corrected when considering ultraviolet completions such as open string theory. The full, α'-exact disk amplitude for the scattering of n gauge bosons has a particularly simple structure [40, 41]

$$\mathcal{A}_n(1\dots n)_{\text{open}} = \sum_{\sigma\in S_{n-3}} F^{\sigma(2\dots n-2)}_{(1\dots n)}(s_{ij};\alpha') A_n\Big(1, \sigma(2,\dots,n-2), n-1, n\Big), \tag{2.4.1}$$

where $A_n(1,\dots,n)$ is the field theory, color ordered gauge amplitude and $F^{\sigma(2\dots n-2)}_{(1\dots n)}(s_{ij};\alpha')$ are generalized Euler integrals over the Koba-Nielsen parameters

$$F^{\sigma(2\dots n-2)}_{(1\dots n)}(s_{ij};\alpha') = (-1)^{n-3} \int\limits_{z_i<z_{i+1}} \left(\prod_{\ell=2}^{n-2} \mathrm{d}z_\ell\right) \prod_{i<j}^{n-1} |z_i - z_j|^{\alpha' s_{ij}} \left\{\prod_{k=2}^{n-2}\left(\sum_{m=1}^{k-1} \frac{\alpha' s_{mk}}{z_m - z_k}\right)\right\}_\sigma , \tag{2.4.2}$$

where the subindex σ indicates that the permutation $\sigma \in S_{n-3}$ acts on all indices inside the curly bracket. These integrals contain the whole α' dependence of $\mathcal{A}_n(1,\dots,n)_{\text{open}}$ and can be seen as a dressing of the gauge theory amplitude to include the effect of the tower of massive string modes.

Similar to the case of Yang-Mills theories, a generic n-point open string amplitude can be expressed in terms of a basis of $(n-3)!$ independent color ordered amplitudes [49]. It is convenient to choose the basis

$$\mathcal{A}_n\Big(1, \Pi_a(2,\dots,n-2), n-1, n\Big)_{\text{open}}, \tag{2.4.3}$$

where $a = 1, \ldots, (n-3)!$ and Π_a denotes the elements of S_{n-3}. Then, Eq. (2.4.1) can be written in matrix form $\boldsymbol{\mathcal{A}}_n = F\mathbf{A}_n$ as

$$\begin{pmatrix} \mathcal{A}_n(1,\Pi_1,n-1,n) \\ \vdots \\ \mathcal{A}_n(1,\Pi_{(n-3)!},n-1,n) \end{pmatrix} = \begin{pmatrix} F_{\Pi_1}{}^{\sigma_1} & \ldots & F_{\Pi_1}{}^{\sigma_{(n-3)!}} \\ \vdots & & \vdots \\ F_{\Pi_{(n-3)!}}{}^{\sigma_1} & \ldots & F_{\Pi_{(n-3)!}}{}^{\sigma_{(n-3)!}} \end{pmatrix} \begin{pmatrix} A_n(1,\sigma_1,n-1,n) \\ \vdots \\ A_n(1,\sigma_{(n-3)!},n-1,n) \end{pmatrix}, \tag{2.4.4}$$

where the shorthand notation $\Pi_a \equiv \Pi_a(2,\ldots,N-2)$ and $\sigma_a \equiv \sigma_a(2,\ldots,N-2)$ has been used.

String corrections to field theory gauge amplitudes are obtained by expanding the integrals (2.4.2) in powers of α'. The coefficients of the series are expressed in terms of kinematic invariants and multiple zeta values (MZV). Thus, the $(n-3)! \times (n-3)!$ matrix F has the following expansion in powers of the inverse string tension [50],

$$\begin{aligned} F = {} & \mathbb{I} + \alpha'^2\zeta(2)P_2 + \alpha'^3\zeta(3)M_3 + \alpha'^4\zeta(2)^2P_4 + \alpha'^5\Big[\zeta(2)\zeta(3)P_2M_3 + \zeta(5)M_5\Big] \\ & + \alpha'^6\left[\zeta(2)^3P_6 + \frac{1}{2}\zeta(3)^2M_3^2\right] + \alpha'^7\Big[\zeta(7)M_7 + \zeta(2)\zeta(5)P_2M_7 + \zeta(2)^2\zeta(3)P_4M_3\Big] \\ & + \ldots \end{aligned} \tag{2.4.5}$$

where

$$M_{2k+1} = F\Big|_{\zeta(2k+1)}, \qquad P_{2k} = F\Big|_{\zeta(2)^k}, \tag{2.4.6}$$

with $P_0 = \mathbb{I}$ and $M_1 = 0$. At order α'^k, the matrix coefficient is a homogeneous function of degree k in the kinematic invariants s_{ij}.

Let us particularize the analysis to the five-point function

$$\begin{pmatrix} \mathcal{A}_5(1,2,3,4,5) \\ \mathcal{A}_5(1,3,2,4,5) \end{pmatrix} = \begin{pmatrix} F_{(12345)}^{(23)} & F_{(12345)}^{(32)} \\ F_{(13245)}^{(23)} & F_{(13245)}^{(32)} \end{pmatrix} \begin{pmatrix} A_5(1,2,3,4,5) \\ A_5(1,3,2,4,5) \end{pmatrix}, \tag{2.4.7}$$

where the matrix entries have the following expansion in powers of the string slope

$$\begin{aligned} F_{(12345)}^{(23)} = {} & 1 + \alpha'^2\zeta(2)(s_{12}s_{34} - s_{34}s_{45} - s_{12}s_{15}) - \alpha'^3\zeta(3)\Big(s_{12}^2s_{34} + 2s_{12}s_{23}s_{34} + s_{12}s_{34}^2 \\ & - s_{34}^2s_{45} - s_{34}s_{45}^2 - s_{12}^2s_{15} - s_{12}s_{15}^2\Big) + \mathcal{O}(\alpha'^4), \end{aligned} \tag{2.4.8}$$

$$F_{(12345)}^{(32)} = \alpha'^2\zeta(2)s_{13}s_{24} - \alpha'^3\zeta(3)s_{13}s_{24}\Big(s_{12} + s_{23} + s_{34} + s_{45} + s_{15}\Big) + \mathcal{O}(\alpha'^4),$$

whereas

$$F^{(23)}_{(13245)} = F^{(32)}_{(12345)}\Big|_{2\leftrightarrow 3} \qquad \text{and} \qquad F^{(32)}_{(13245)} = F^{(23)}_{(12345)}\Big|_{2\leftrightarrow 3}. \tag{2.4.9}$$

Writing the kinematic invariants in (2.4.8) in terms of the stereographic coordinates, we see that the expansion parameter is $s\alpha' \ll 1$. This can be traced back to Eq. (2.4.2), where all dependence on α' comes through the dimensionless combination $s_{ij}\alpha' = (s\alpha')f_{ij}(\zeta_a)$, with f_{ij} a function of the stereographic coordinates.

We compute next the full color-dressed five-point disk amplitude. Following Ch. 1, we work in the Yang-Mills amplitude basis $A_5(1, \sigma(3, 4, 5), 2)$, which means that we use the Jacobi identities to recast all color factors in terms of $\{c_2, c_6, c_7, c_8, c_{11}, c_{13}\}$. Namely,

$$\begin{aligned}\mathcal{A}_{5,\text{string}} &= c_2\mathcal{A}_5(1,5,4,3,2)_{\text{open}} + c_6\mathcal{A}_5(1,4,3,5,2)_{\text{open}} + c_7\mathcal{A}_5(1,3,4,5,2)_{\text{open}} \\ &\quad + c_8\mathcal{A}_5(1,3,5,4,2)_{\text{open}} + c_{11}\mathcal{A}_5(1,5,3,4,2)_{\text{open}} + c_{13}\mathcal{A}_5(1,4,5,3,2)_{\text{open}}.\end{aligned} \tag{2.4.10}$$

Using now Eq. (2.4.7), the full string amplitudes on the right-hand side of this equation are expressed in terms of our basis of color-ordered Yang-Mills amplitudes as

$$\begin{aligned}&\mathcal{A}_{5,\text{string}} \\ &= \left(c_2F^{(54)}_{(15432)} + c_{13}F^{(54)}_{(14532)}\right)A_5(1,5,4,3,2) + \left(c_6F^{(43)}_{(14352)} + c_7F^{(43)}_{(13452)}\right)A_5(1,4,3,5,2) \\ &+ \left(c_6F^{(34)}_{(14352)} + c_7F^{(34)}_{(13452)}\right)A_5(1,3,4,5,2) + \left(c_8F^{(35)}_{(13542)} + c_{11}F^{(35)}_{(15342)}\right)A_5(1,3,5,4,2) \\ &+ \left(c_8F^{(53)}_{(13542)} + c_{11}F^{(53)}_{(15342)}\right)A_5(1,5,4,3,2) + \left(c_2F^{(45)}_{(15432)} + c_{13}F^{(45)}_{(14532)}\right)A_5(1,4,5,3,2).\end{aligned} \tag{2.4.11}$$

Finally, we use the expressions for the color subamplitudes given by the Parke-Taylor formula[2] [13] and implement the expansions (2.4.8) and (2.4.9). Using the stereographic coordinates defined in (2.1.3) and (2.1.4), we arrive at the final expression for the five-point disk amplitude at order α'^3 in the planar limit:

$$\begin{aligned}\mathcal{A}_{5,\text{string}}\Big|_{\text{planar}} &= \frac{i(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)}{\sqrt{s}\zeta_3\zeta_4\zeta_5(1+\zeta_3\zeta_4)(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)}\left[A_5^{(0)} + \frac{(s\alpha')^2\zeta(2)A_5^{(2)}}{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)}\right. \\ &\quad \left. + \frac{(s\alpha')^3\zeta(3)A_5^{(3)}}{(\zeta_3-\zeta_4)^2(\zeta_3-\zeta_5)^2(\zeta_4-\zeta_5)^2} + \mathcal{O}\Big((s\alpha')^4\Big)\right].\end{aligned} \tag{2.4.12}$$

The coefficient $A_5^{(0)}$ is the cubic homogeneous polynomial determining the planar zeros

[2] As in Ch. 1, we consider MHV amplitudes with helicities $(1^-, 2^-, 3^+, 4^+, 5^+)$.

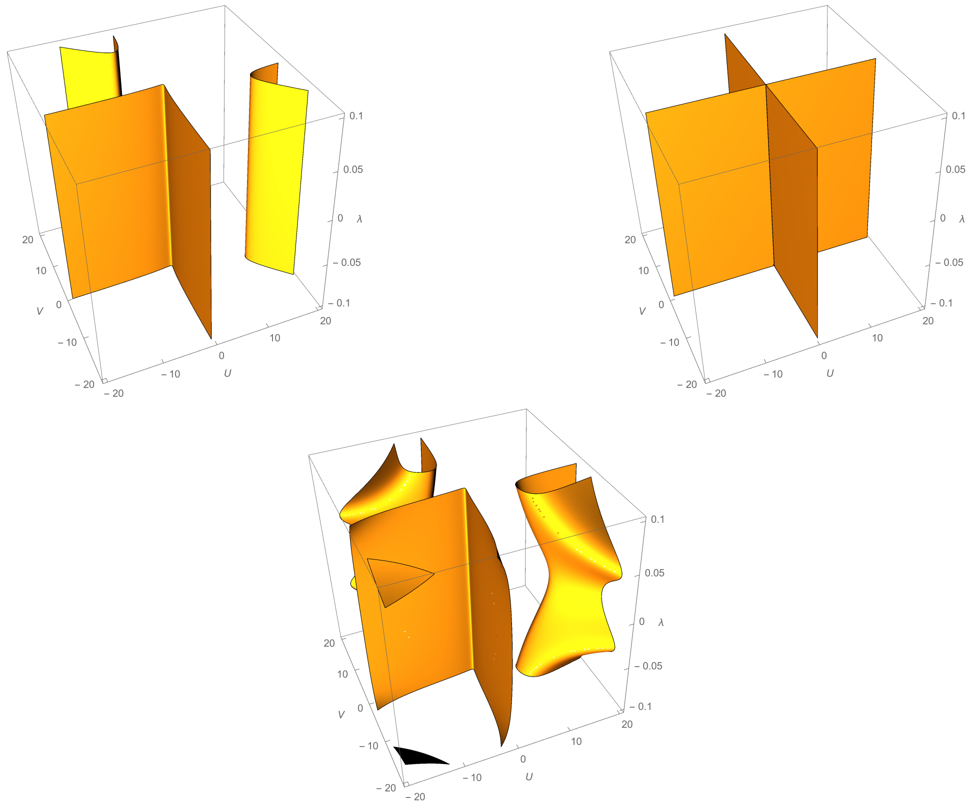


Figure 2.2: Surfaces in the projective space (λ, U, V) associated to planar zeros for incoming open strings in a singlet state. (Left) Field theory limit $\mathcal{A}^{(0)} = 0$ showing the projective nature of planar zeros. See Fig. 1.4. (Right) α'^2-correction $\mathcal{A}^{(2)} = 0$ showing projectivity although ouside the physical region. (Below) α'^3-correction $\mathcal{A}^{(3)} = 0$ losing the projective structure.

of the five-gluon amplitude

$$\begin{aligned} A_5^{(0)}(\zeta_3, \zeta_4, \zeta_5) &= c_7\zeta_3^2\zeta_4 - c_8\zeta_3^2\zeta_5 - c_6\zeta_3\zeta_4^2 + c_{11}\zeta_3\zeta_5^2 \\ &\quad + (c_2 + c_6 - c_7 + c_8 - c_{11} + c_{13})\zeta_3\zeta_4\zeta_5 + c_{13}\zeta_4^2\zeta_5 - c_2\zeta_4\zeta_5^2. \end{aligned} \tag{2.4.13}$$

However, the α'^2 and α'^3 coefficients $A_5^{(2)}$ and $A_5^{(3)}$ are respectively degree 10 and 15, *non-homogeneus* polynomials whose explicit expressions are given in Eqs. (B.2.1) and (B.2.2) of the Appendix. Thus, α' corrections destroy the projective properties of the loci of planar zeros found in Ch. 1. Interestingly, when considering the scattering of two gluons in a singlet state

$$c_2 = c_6 = -c_7 = c_8 = -c_{11} = -c_{13} = -f^{a_3a_4a_5}, \tag{2.4.14}$$

the equation $A_5^{(2)} = 0$ becomes a homogeneous polynomial

$$\zeta_3\zeta_4\zeta_5(\zeta_3 - \zeta_4)(\zeta_3 - \zeta_5)(\zeta_4 - \zeta_5) = 0. \tag{2.4.15}$$

However, the zeros of this equation all lie at unphysical values of the stereographic coordinates for which either the amplitude or the energy of at least one of the outgoing particles diverges. Check the behavior in Fig. 2.2.

2.5 Gravitational amplitudes

One of the results of the previous chapter is that the planar, MHV five-point graviton amplitude is identically zero. This fact can be seen as a consequence of the theorem proved in [39], stating the vanishing of all helicity violating amplitudes in three dimensions. Indeed, at the level of the tree amplitude, the graviton couplings are of the form $p_i \cdot \varepsilon_k \cdot p_j$ with $i, j \neq k$, so imposing planarity decouples the graviton polarization normal to the plane. This renders the scattering effectively three-dimensional and, as a consequence, the planar MHV amplitude is equal to zero.

In this section we are going to explore other gravitational amplitudes involving scalar particles minimally coupled to gravity. We begin with the scattering of two distinguishable scalars with graviton emission

$$\Phi(p_1) + \Phi'(p_2) \longrightarrow \Phi(p_3) + \Phi'(p_4) + G(p_5, \varepsilon). \tag{2.5.1}$$

The tree-level amplitude was computed in Ref. [51] using the Feynman rules for a scalar theory coupled to gravity. Using the Sudakov decomposition,

$$\begin{aligned} k_1 &\equiv -p_1 - p_3 = \alpha_1 p_1 + \beta_1 p_2 + k_{1,\perp}, \\ k_2 &\equiv -p_2 - p_4 = \alpha_2 p_1 + \beta_2 p_2 + k_{2,\perp}, \end{aligned} \tag{2.5.2}$$

the amplitude has the tensor structure

$$\begin{aligned} \mathcal{M} = \left(\frac{\kappa}{2}\right)^3 \bigg\{ &(K_\perp \cdot \varepsilon \cdot K_\perp) A_{KK} + \Big[(K_\perp \cdot \varepsilon \cdot p_1) + (p_1 \cdot \varepsilon \cdot K_\perp)\Big] A_{K1} \\ &+ \Big[(K_\perp \cdot \varepsilon \cdot p_2) + (p_2 \cdot \varepsilon \cdot K_\perp)\Big] A_{K2} + (p_1 \cdot \varepsilon \cdot p_1) A_{11} + (p_2 \cdot \varepsilon \cdot p_2) A_{22} \\ &+ \Big[(p_1 \cdot \varepsilon \cdot p_2) + (p_2 \cdot \varepsilon \cdot p_1)\Big] A_{12} \bigg\}, \end{aligned} \tag{2.5.3}$$

where $K_\perp \equiv k_{1,\perp} + k_{2,\perp}$ and the coefficients A_i are rational functions of the Sudakov parameters α_i, β_i. The tensor structure of the amplitude shows again how, once the planar limit $\zeta_i = \overline{\zeta}_i$ is taken, the polarizations outside the interaction plane decouple and the amplitude becomes effectively three-dimensional. In this limit, the Sudakov parameters take the following form in terms of the stereographic coordinates:

$$\begin{aligned}
\alpha_1 &\equiv \frac{p_2\cdot(p_1+p_3)}{p_1\cdot p_2} = \frac{\zeta_4\zeta_5(1-\zeta_3^2)-\zeta_3(\zeta_4+\zeta_5)}{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)},\\
\beta_1 &\equiv \frac{p_1\cdot(p_1+p_3)}{p_1\cdot p_2} = -\frac{1+\zeta_4\zeta_5}{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)},\\
\alpha_2 &\equiv -\frac{p_2\cdot(p_2+p_4)}{p_1\cdot p_2} = -\frac{\zeta_4^2(1+\zeta_3\zeta_5)}{(\zeta_3-\zeta_4)(\zeta_4-\zeta_5)},\\
\beta_2 &\equiv -\frac{p_1\cdot(p_2+p_4)}{p_1\cdot p_2} = \frac{-1+\zeta_4(-\zeta_3+\zeta_4-\zeta_5)}{(\zeta_3-\zeta_4)(\zeta_4-\zeta_5)},
\end{aligned} \tag{2.5.4}$$

while the graviton polarization tensor is taken to be $\varepsilon_\pm = \epsilon_\pm \otimes \epsilon_\pm$, with $\epsilon_\pm$ defined by (2.3.6). Using the explicit expression for the coefficients in (2.5.3) given in [51], we find that the planar amplitude vanishes identically

$$\mathcal{M}\Big|_{\text{planar}} = 0. \tag{2.5.5}$$

It was found in [51] that this gravitational amplitude can be split into two gauge invariant subamplitudes, $\mathcal{M} = \mathcal{M}_\uparrow + \mathcal{M}_\downarrow$, where each term can be written in terms of an effective, nonlocal vertex. In the planar limit, these subamplitudes are individually nonzero and take a specially simple form

$$\mathcal{M}_\uparrow\Big|_{\text{planar}} = -\mathcal{M}_\downarrow\Big|_{\text{planar}} = \left(\frac{\kappa s}{4}\right)\frac{\zeta_3(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)}{(1+\zeta_3\zeta_4)(\zeta_4-\zeta_5+\zeta_3\zeta_4\zeta_5-\zeta_4\zeta_5^2)}. \tag{2.5.6}$$

The gravitational amplitude (2.5.3) cannot be retrieved using the double-copy BCJ construction [12, 15] from the gauge scattering amplitude of two distinct scalars with a gluon emission [46]. Despite this, the putative gravitational amplitude obtained from the double-copy of the gauge amplitude (2.3.2), with denominators satisfying color-kinematics duality, identically vanishes in the planar limit.

A similar result is obtained for the gravitational scattering of two indistinguishable scalars. In this case, the amplitude can be obtained by double copy from the gauge scattering of adjoint identical scalars given in Eq. (2.3.22) using color-kinematics duality [48],

$$\begin{aligned}
\mathcal{M}_5 = \left(\frac{\kappa}{2}\right)^3 \bigg(&\frac{n_1^2}{s_{12}s_{45}} + \frac{n_2^2}{s_{23}s_{15}} + \frac{n_3^2}{s_{34}s_{12}} + \frac{n_4^2}{s_{45}s_{23}} + \frac{n_5^2}{s_{15}s_{34}} + \frac{n_6^2}{s_{14}s_{25}} + \frac{n_7^2}{s_{13}s_{25}} + \frac{n_8^2}{s_{24}s_{13}}\\
&+ \frac{n_9^2}{s_{35}s_{24}} + \frac{n_{10}^2}{s_{14}s_{35}} + \frac{n_{11}^2}{s_{15}s_{24}} + \frac{n_{12}^2}{s_{12}s_{35}} + \frac{n_{13}^2}{s_{23}s_{14}} + \frac{n_{14}^2}{s_{25}s_{34}} + \frac{n_{15}^2}{s_{13}s_{45}} \bigg),
\end{aligned} \tag{2.5.7}$$

where the numerators are the ones given in Eq. (2.3.23). In fact, the cancellation of this amplitude in the planar limit can be seen to happen by a mechanism similar to the one found in Eq. (1.5.9) for the pure gravitational case. Indeed, replacing the color factors by the corresponding numerators in the condition for the gauge planar zeros (2.3.25), we find the following condition for the existence of planar zeros

$$\begin{aligned} &n_7\zeta_3^2\zeta_4 - n_8\zeta_3^2\zeta_5 - n_6\zeta_3\zeta_4^2 + n_{11}\zeta_3\zeta_5^2 \\ &\qquad +(n_2 + n_6 - n_7 + n_8 - n_{11} - n_{13})\zeta_3\zeta_4\zeta_5 + n_{13}\zeta_4^2\zeta_5 - n_2\zeta_4\zeta_5^2 = 0. \end{aligned} \tag{2.5.8}$$

In the planar limit (i.e., real stereographic coordinates), the relevant numerators have the following form

$$\begin{aligned} n_2 &= \frac{s^{\frac{3}{2}}}{\sqrt{2}(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)(1+\zeta_5^2)}\Big(1 + \zeta_3\zeta_4 - 2\zeta_3\zeta_5 + 5\zeta_4\zeta_5 - 2\zeta_3^2\zeta_5^2 \\ &\quad + \zeta_3\zeta_4\zeta_5^2 + 4\zeta_4^2\zeta_5^2 - \zeta_3^2\zeta_4^2\zeta_5^2 - 2\zeta_3^2\zeta_4\zeta_5^3 + 4\zeta_3\zeta_4^2\zeta_5^3\Big), \\ n_6 &= \frac{s^{\frac{3}{2}}\zeta_5}{\sqrt{2}(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)(1+\zeta_5^2)}\Big(- 2\zeta_3 + 4\zeta_4 - \zeta_5 - 2\zeta_3^2\zeta_5 + \zeta_3\zeta_4\zeta_5 \\ &\quad + 4\zeta_4^2\zeta_5 - 2\zeta_3^2\zeta_4\zeta_5^2 + 4\zeta_3\zeta_4^2\zeta_5^2 + \zeta_3\zeta_4\zeta_5^3 + \zeta_3^2\zeta_4^2\zeta_5^3\Big), \\ n_7 &= -\frac{s^{\frac{3}{2}}\zeta_5}{\sqrt{2}(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)(1+\zeta_5^2)}(4\zeta_3 - 2\zeta_4 - \zeta_5 + 4\zeta_3^2\zeta_5 + \zeta_3\zeta_4\zeta_5 \\ &\quad - 2\zeta_4^2\zeta_5 + 4\zeta_3^2\zeta_4\zeta_5^2 - 2\zeta_3\zeta_4^2\zeta_5^2 + \zeta_3\zeta_4\zeta_5^3 + \zeta_3^2\zeta_4^2\zeta_5^3\Big), \\ n_8 &= \frac{s^{\frac{3}{2}}}{\sqrt{2}(\zeta_3-\zeta_4)^2(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)(1+\zeta_5^2)}\Big(\zeta_4 - \zeta_5 - \zeta_3^3\zeta_4^3\zeta_5^3 - 4\zeta_3^3\zeta_4^2\zeta_5^2 \\ &\quad - 4\zeta_3^3\zeta_4\zeta_5 + \zeta_3^2\zeta_4^3\zeta_5^4 + \zeta_3^2\zeta_4^3\zeta_5^2 + 8\zeta_3^2\zeta_4^2\zeta_5^3 - \zeta_3^2\zeta_4^2\zeta_5 - 4\zeta_3^2\zeta_4\zeta_5^4 + 8\zeta_3^2\zeta_4\zeta_5^2 \\ &\quad - 4\zeta_3^2\zeta_4 - 4\zeta_3^2\zeta_5^3 - 4\zeta_3\zeta_4^3\zeta_5^3 + 2\zeta_3\zeta_4^3\zeta_5 + 3\zeta_3\zeta_4^2\zeta_5^4 + 2\zeta_3\zeta_4^2\zeta_5^2 + 3\zeta_3\zeta_4^2 \\ &\quad - \zeta_3\zeta_4\zeta_5^3 + 8\zeta_3\zeta_4\zeta_5 - 4\zeta_3\zeta_5^2 - 4\zeta_4^3\zeta_5^2 + 2\zeta_4^2\zeta_5^3 - 4\zeta_4^2\zeta_5 + \zeta_4\zeta_5^2\Big), \\ n_{11} &= \frac{s^{\frac{3}{2}}}{\sqrt{2}(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)(1+\zeta_5^2)}\Big(- 1 - \zeta_3\zeta_4 - 4\zeta_3\zeta_5 - 4\zeta_3^2\zeta_5^2 \\ &\quad - \zeta_3\zeta_4\zeta_5^2 + 2\zeta_4^2\zeta_5^2 + \zeta_3^2\zeta_4^2\zeta_5^2 - 4\zeta_3^2\zeta_4\zeta_5^3 + 2\zeta_3\zeta_4^2\zeta_5^3\Big), \\ n_{13} &= \frac{s^{\frac{3}{2}}}{\sqrt{2}(\zeta_3-\zeta_4)^2(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)(1+\zeta_5^2)}\Big(\zeta_3 - \zeta_5 - \zeta_3^3\zeta_4^3\zeta_5^3 + \zeta_3^3\zeta_4^2\zeta_5^4 \\ &\quad + \zeta_3^3\zeta_4^2\zeta_5^2 - 4\zeta_3^3\zeta_4\zeta_5^3 + 2\zeta_3^3\zeta_4\zeta_5 - 4\zeta_3^3\zeta_5^2 - 4\zeta_3^2\zeta_4^3\zeta_5^2 + 8\zeta_3^2\zeta_4^2\zeta_5^3 - \zeta_3^2\zeta_4^2\zeta_5 \\ &\quad + 3\zeta_3^2\zeta_4\zeta_5^4 + 2\zeta_3^2\zeta_4\zeta_5^2 + 3\zeta_3^2\zeta_4 + 2\zeta_3^2\zeta_5^3 - 4\zeta_3^2\zeta_5 - 4\zeta_3\zeta_4^3\zeta_5 - 4\zeta_3\zeta_4^2\zeta_5^4 \end{aligned} \tag{2.5.9}$$

$$+ 8\zeta_3\zeta_4^2\zeta_5^2 - 4\zeta_3\zeta_4^2 - \zeta_3\zeta_4\zeta_5^3 + 8\zeta_3\zeta_4\zeta_5 + \zeta_3\zeta_5^2 - 4\zeta_4^2\zeta_5^3 - 4\zeta_4\zeta_5^2\Big).$$

Substituting these values in Eq. (2.5.8), we conclude that the condition for the existence of planar zeros is identically satisfied for any kinematic configuration. Since the numerators now are far more complicated than the ones for pure gluon scattering, the cancellation taking place is less trivial.

Since the scalar gravitational amplitudes studied above do not preserve helicity, the fact that they are zero in the planar limit is also a consequence of the vanishing of all helicity-violating supergravity amplitudes when reduced to three dimensions [39] (see also [5]). Indeed, the gauge amplitude for indistinguishable adjoint scalars (2.3.22) can be embedded in a $\mathcal{N} = 2$ super Yang-Mills theory [48]. Thus, the corresponding double copy can be thought of as a scattering amplitude in $\mathcal{N} = 4$ supergravity [52]. In the case of the gravitational scattering of two distinct scalars (2.5.1), on the other hand, the theory can be also embedded in a four-dimensional supergravity theory, such as the ones studied in [52]. Both amplitudes vanish in the planar limit, where the dynamics becomes effectively three-dimensional.

In the case of graviton MHV amplitudes, their vanishing in the planar limit follows from the explicit expression of the n-graviton amplitude [53, 54]

$$\begin{aligned} M_n^{\text{MHV}} = &\sum_{P(1,\ldots,n-3)} \frac{1}{\langle n\, n-2\rangle\langle n-2\, n-1\rangle\langle n-1\, n\rangle}\frac{1}{\langle 12\rangle\ldots\langle n1\rangle} \\ &\times \prod_{k=1}^{n-3}\frac{[k|p_{k+1}+\ldots+p_{n-2}|n-1\rangle}{\langle k\, n-1\rangle}, \end{aligned} \tag{2.5.10}$$

where the sum runs over all permutation of the labels $1, \ldots, n-3$ and the notation

$$[a|p_{k_1}+\ldots+p_{k_n}|b\rangle \equiv [ak_1]\langle k_1 b\rangle + \ldots + [ak_n]\langle k_n b\rangle, \tag{2.5.11}$$

has been used. Using this expression, we have explicitly checked that

$$M_n^{\text{MHV}}\Big|_{\text{planar}} = 0, \qquad \text{for } n = 5, 6, 7, \text{ and } 8, \tag{2.5.12}$$

as expected.

For the scattering of four gravitons, Eq. (2.5.10) gives the helicity preserving amplitude $M_4(1^+, 2^+, 3^-, 4^-)$. Notice that the four-point amplitude is always planar, and the result obtained by applying (2.5.10) is however different from zero

$$M_4\Big|_{\text{planar}} \equiv M_4 = s\left(\frac{1+\zeta_3^2}{\zeta_3}\right)^2, \tag{2.5.13}$$

where we have chosen coordinates such that the process takes place on the plane $y = 0$ (i.e., $\zeta_3 = -\zeta_4^{-1} \in \mathbb{R}$). Another scattering amplitude whose vanishing in the planar

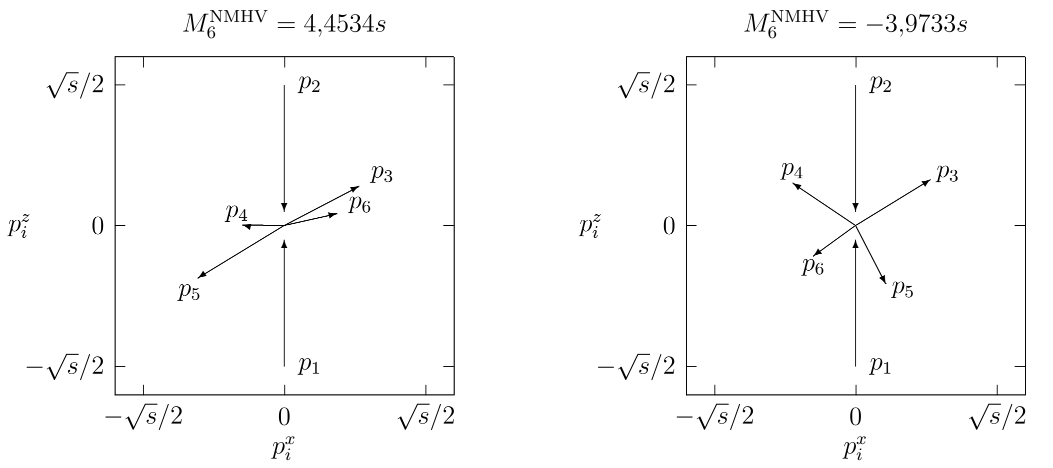


Figure 2.3: Two kinematic configurations for which the planar, helicity preserving six-graviton amplitude in Eq. (2.5.14) is nonzero.

limit is not implied by the results of [39] is the six-graviton, helicity preserving amplitude $M_6(1^+,2^-,3^-,4^-,5^+,6^+)$. This can be computed starting with the six-gluon, helicity preserving amplitude [4],

$$\begin{aligned} A_6(1^+,2^-,3^-,4^-,5^+,6^+) = {} & \frac{\langle 4|p_2+p_3|1]^3}{(p_1+p_2+p_3)^2[12][23]\langle 45\rangle\langle 56\rangle\langle 6|p_1+p_2|3]} \\ & + \frac{\langle 2|p_1+p_6|5]^3}{(p_1+p_2+p_6)^2[34][45]\langle 61\rangle\langle 12\rangle\langle 6|p_1+p_2|3]}, \end{aligned} \tag{2.5.14}$$

and applying the KLT formula

$$M_6 = -\kappa^4 \boldsymbol{A}_6^T S_0 \boldsymbol{A}_6 \tag{2.5.15}$$

where S_0 is the field theory KLT kernel introduced in Eq. (2.6.5).

The explicit expression for the amplitude $M_6(1^+,2^-,3^-,4^-,5^+,6^+)$ in the planar limit in terms of the stereographic coordinates is very cumbersome and will not be given here. However, it can be seen that this amplitude does not vanish. In Fig. 2.3 we have depicted two kinematic *planar* configurations for which a calculation of the tree-level amplitude gives a nonzero result.

2.6 String corrections to graviton planar scattering

String graviton amplitudes on the sphere can be written in terms of disk amplitudes of gauge bosons using the Kawai-Lewellen-Tye (KLT) relations [17]. A general expression for the n-gravity amplitude reads [18]

$$\begin{aligned}\mathcal{M}_n = (-1)^{n-3}\kappa^{n-2} &\sum_{\sigma,\rho\in S_{n-3}} \mathcal{A}_n\Big(1,\sigma(2,\ldots,n-2),n-1,n\Big)\\ &\times \mathcal{S}[\rho|\sigma]_1 \widetilde{\mathcal{A}}_n\Big(1,\rho(2,\ldots,n-2),n,n-1\Big),\end{aligned} \tag{2.6.1}$$

where the two gauge copies differ by the ordering of the last two entries. The momentum kernel $\mathcal{S}[\rho|\sigma]_1$ has the form

$$\begin{aligned}\mathcal{S}[\rho|\sigma]_1 &\equiv \mathcal{S}[\rho(2,\ldots,n-2)|\sigma(2,\ldots,n-2)]_1\\ &= \left(\frac{2}{\pi\alpha'}\right)^{n-3}\prod_{j=2}^{n-2}\sin\left[\frac{\pi\alpha'}{2}\left(s_{1j_p}+\sum_{k=2}^{j-1}\theta(j_\rho,k_\rho)s_{j_\rho k_\rho}\right)\right],\end{aligned} \tag{2.6.2}$$

where the symbol $\theta(j_\rho,k_\rho)$ equals 1 if the legs j_ρ and k_ρ keep the same order in the sets $\rho(2,\ldots,n-2)$ and $\sigma(2,\ldots,n-2)$, and 0 otherwise.

The generalized KLT relations (2.6.1) can be recast in matrix form as [44]

$$\mathcal{M}_n = (-1)^{n-3}\kappa^{n-2}\widetilde{\boldsymbol{\mathcal{A}}}_n^T\mathcal{S}\boldsymbol{\mathcal{A}}_n = (-1)^{n-3}\kappa^{n-2}\boldsymbol{\mathcal{A}}_n^T\mathcal{S}_0\boldsymbol{\mathcal{A}}_n, \tag{2.6.3}$$

where in the second identity we have changed the basis of the first-copy amplitudes to express them in terms of of the basis $\boldsymbol{\mathcal{A}}_n$ used in Eq. (2.4.4). Using now this same equation, we can express the string graviton amplitude in terms of field theory gauge amplitudes as

$$\mathcal{M}_n = (-1)^{n-3}\kappa^{n-2}\boldsymbol{A}_n^T F^T\mathcal{S}_0 F\boldsymbol{A}_n. \tag{2.6.4}$$

The single-valued projection [42–44, 55] allows a further simplification of this relation. It projects the MZVs appearing in the expansion of the matrix F in (2.4.5) to a subclass $\mathrm{sv}(F)$, called the single-valued MZVs, which exactly reproduces the closed string α' expansion

$$F^T\mathcal{S}_0F = S_0\,\mathrm{sv}(F), \tag{2.6.5}$$

where S_0 is the field theory limit ($\alpha'\to 0$) KLT kernel in the basis $\boldsymbol{\mathcal{A}}_n$. The action of the single-valued projection on the MZVs is given by

$$\begin{aligned}\mathrm{sv}[\zeta(2)] &= 0,\\ \mathrm{sv}[\zeta(2n+1)] &= 2\zeta(2n+1), \qquad \text{for } n\geq 1.\end{aligned} \tag{2.6.6}$$

With this, the string amplitude takes the form

$$\mathcal{M}_n = (-1)^{n-3}\kappa^{n-2}\boldsymbol{A}_n^T S_0\,\mathrm{sv}(F)\boldsymbol{A}_n. \tag{2.6.7}$$

Incidentally, dropping the term $\mathrm{sv}(F)$ in the previous expression we retrieve the KLT expression of the field theory graviton amplitude.

We particularize our analysis to the five-point amplitude

$$\mathcal{M}_5 = \kappa^3\boldsymbol{A}_5^T S_0\,\mathrm{sv}(F)\boldsymbol{A}_5, \tag{2.6.8}$$

where, from (2.4.5), we have

$$\begin{aligned}\mathrm{sv}(F) = \mathbb{I} &+ 2\left(\frac{\alpha'}{4}\right)^3\zeta(3)M_3 + 2\left(\frac{\alpha'}{4}\right)^5\zeta(5)M_5\\ &+ 2\left(\frac{\alpha'}{4}\right)^6\zeta(3)^2M_3^2 + 2\left(\frac{\alpha'}{4}\right)^7\zeta(7)M_7 + \dots\end{aligned} \tag{2.6.9}$$

In order to get the closed string expression, we have to perform the rescaling $\alpha' \to \alpha'/4$, as explained in [42]. Notice that the single-valued projection (2.6.6) eliminates many terms in the α'-expansion of F. Plugging (2.6.9) into Eq. (2.6.7), we see how the first term gives, via the KLT relations, the field theory gravity amplitude, while the second one corresponds to the first nonvanishing string correction. The entries of the matrix M_3 can be read from Eqs. (2.4.8) and (2.4.9). The matrix S_0 is given by $S_0 = K^T S$, where S is the $\alpha' \to 0$ limit of the KLT kernel in Eq. (2.6.2) and K implements the change of basis $\widetilde{\boldsymbol{A}}_5 = K\boldsymbol{A}_5$ in the first copy. Using the KK and BCJ relations[3], this matrix is given by

$$K = \begin{pmatrix} \frac{s_{34}(s_{35}-s_{24})}{s_{14}s_{35}} & \frac{s_{13}s_{24}}{s_{14}s_{35}} \\ \frac{s_{12}s_{34}}{s_{14}s_{25}} & \frac{s_{24}(s_{25}-s_{34})}{s_{14}s_{25}} \end{pmatrix}. \tag{2.6.10}$$

Imposing the planarity condition in the stereographic coordinates, $\zeta_a \in \mathbb{R}$, we confirm the result of Ch. 1

$$\kappa^3\boldsymbol{A}_5^T S_0\boldsymbol{A}_5\Big|_{\text{planar}} = 0. \tag{2.6.11}$$

However, a first nonvanishing string correction survives the planar limit,

$$\mathcal{M}_5\Big|_{\text{planar}} = \frac{3\zeta(3)}{32}\alpha'^3\kappa^3 s^4 + \mathcal{O}(\alpha'^5). \tag{2.6.12}$$

[3]See the Preface for details.

This term is independent of the directions of the final states and is never zero. Using the expansion (2.4.5), it is possible to compute higher order corrections, whose coefficients are functions of the stereographic coordinates ζ_a. We obtain the structure

$$
\begin{aligned}
\mathcal{M}_5\Big|_{\text{planar}} &= \frac{3\zeta(3)}{32}\alpha'^3\kappa^3 s^4 + \frac{5\zeta(5)}{512}\alpha'^5\kappa^3 s^6 \frac{Q_{10}(\zeta_3,\zeta_4,\zeta_5)}{(\zeta_3-\zeta_4)^2(\zeta_3-\zeta_5)^2(\zeta_4-\zeta_5)^2} \\
&- \frac{3\zeta(3)^2}{2048}\alpha'^6\kappa^3 s^7 \frac{Q_{12}(\zeta_3,\zeta_4,\zeta_5)}{(\zeta_3-\zeta_4)^2(\zeta_3-\zeta_5)^2(\zeta_4-\zeta_5)^2} \\
&+ \frac{7\zeta(7)}{8192}\alpha'^7\kappa^3 s^8 \frac{Q_{10}(\zeta_3,\zeta_4,\zeta_5)^2}{(\zeta_3-\zeta_4)^4(\zeta_3-\zeta_5)^4(\zeta_4-\zeta_5)^4} \\
&- \frac{\zeta(3)\zeta(5)}{32768}\alpha'^8\kappa^3 s^9 \frac{Q_{22}(\zeta_3,\zeta_4,\zeta_5)}{(\zeta_3-\zeta_4)^4(\zeta_3-\zeta_5)^4(\zeta_4-\zeta_5)^4} + \ldots
\end{aligned}
\tag{2.6.13}
$$

The numerators $Q_n(\zeta_3,\zeta_4,\zeta_5)$ appearing in this expansion are *non-homogeneous* polynomials of degree n whose explicit expressions are given in Eqs. (B.3.1)-(B.3.3) of the Appendix. Our results show how the exchange of massive string modes renders the planar gravitational amplitude nonzero, with the higher order terms in the α' expansion determined by nonhomogeneous polynomials.

It is interesting to notice that the planar closed string amplitude (2.6.13) does not exhibit the soft poles at $\zeta_a\zeta_b = -1$ (with $a < b$), unlike the planar disk amplitude in Eq. (2.4.12). This reflects the peculiar relation between the soft and planar limits of amplitudes with gravitons, in both string and field theories. It would be worthwhile to clarify the interplay between the two limits using recent results for soft theorems in string theory [56, 57].

2.7 Remarks on soft limits

We turn now to the problem of whether the mathematical structure of planar zeros can be fully captured in the soft limit. We begin with the gauge case analyzing the simple example of two distinguishable scalars studied in Section 2.3.1. In the limit in which the emitted positive (resp. negative) helicity gluon is soft, $p_5 \to 0$, the leading behavior of the amplitude takes the form [21]

$$
\begin{aligned}
\mathcal{A}_{5,\text{soft}} &= 2g\left(C_1\frac{p_3\cdot\epsilon_\pm}{s_{35}} - C_2\frac{p_1\cdot\epsilon_\pm}{s_{15}} + C_4\frac{p_4\cdot\epsilon_\pm}{s_{45}} - C_5\frac{p_2\cdot\epsilon_\pm}{s_{25}}\right)\mathcal{A}_4 \\
&= 2g\left[C_1\left(\frac{p_3\cdot\epsilon_\pm}{s_{35}} - \frac{p_2\cdot\epsilon_\pm}{s_{25}}\right) + C_2\left(\frac{p_2\cdot\epsilon_\pm}{s_{25}} - \frac{p_1\cdot\epsilon_\pm}{s_{15}}\right) + C_4\left(\frac{p_4\cdot\epsilon_\pm}{s_{45}} - \frac{p_2\cdot\epsilon_\pm}{s_{25}}\right)\right]\mathcal{A}_4,
\end{aligned}
\tag{2.7.1}
$$

where $\mathcal{A}_4$ is the four-scalar tree level amplitude. In terms of the stereographic coordinates ζ_a and taking the planar scattering limit, the soft amplitude reads

$$\begin{aligned}\mathcal{A}_{5,\text{soft}}\Big|_{\text{planar}} &= \mp\frac{g\sqrt{2}}{\sqrt{s}\zeta_5(1+\zeta_3\zeta_4)} \\ &\times\Big[(C_1-C_2+C_4)\zeta_3\zeta_4-(C_1-C_2)\zeta_3\zeta_5+C_2-C_4)\zeta_4\zeta_5-C_2\zeta_5^2\Big]\mathcal{A}_4. \quad (2.7.2)\end{aligned}$$

The condition for the vanishing of the soft gauge theory amplitude in the planar limit is given by

$$(C_1-C_2+C_4)\zeta_3\zeta_4-(C_1-C_2)\zeta_3\zeta_5+(C_2-C_4)\zeta_4\zeta_5-C_2\zeta_5^2=0, \quad (2.7.3)$$

which reproduces the nontrivial loci of planar zeros for the full tree level amplitude discussed in Eq. (2.3.11). We notice, however, that in taking the soft limit we miss the trivial branch $2\zeta_3-\zeta_4=0$. In fact, this loci cannot be captured in the soft-gluon limit of the amplitude, since in the limit $\omega_5\to 0$,

$$1+\zeta_3\zeta_4\longrightarrow 0 \qquad\Longrightarrow\qquad \zeta_4\longrightarrow -\frac{1}{\zeta_3}, \quad (2.7.4)$$

so we have

$$2\zeta_3-\zeta_4\longrightarrow\frac{2\zeta_3^2+1}{\zeta_3}, \quad (2.7.5)$$

which implies that $2\zeta_3-\zeta_4$ never vanishes. This shows that the trivial branch of planar zeros is not accesible from the soft limit of the amplitude. Therefore, not all planar zeros can be realized in the limit in which the gluon is taken to be soft. Notice, however, that this does not contradict the statements made in the previous chapter. Indeed, any planar zero can be realized in the limit in which *one* of the particles is taken to be soft. However, once we decide which particle is soft, not all planar zeros can be realized in this regime, as we have seen in this case.

This being said, soft limits can be exploited to make a general analysis of planar zeros in the gauge case. We study the scattering of n charged particles in QED, parametrized by stereographic coordinates ζ_i $(i=1,\ldots,n)$, with the emission of a soft photon whose momenta we write in terms of the coordinate ζ_{n+1},

$$p_a = \omega_a\left(1,\frac{\zeta_a+\overline{\zeta}_a}{1+\zeta_a\overline{\zeta}_a},i\frac{\overline{\zeta}_a-\zeta_a}{1+\zeta_a\overline{\zeta}_a},\frac{\zeta_a\overline{\zeta}_a-1}{1+\zeta_a\overline{\zeta}_a}\right), \qquad a=1,\ldots,n+1. \quad (2.7.6)$$

The soft theorem for massless QED can be recast in terms of stereographic coordinates as [58]

$$\lim_{\omega_{n+1}\to 0^+}\Big[\omega_{n+1}\mathcal{A}_{n+1}(p_1,\ldots,p_{n+1})\Big]$$

$$=\frac{1}{\sqrt{2}}(1+\zeta_{n+1}\bar{\zeta}_{n+1})\left(\sum_{i\in\text{out}}\frac{e_i}{\zeta_i-\zeta_{n+1}}-\sum_{j\in\text{in}}\frac{e_j}{\zeta_j-\zeta_{n+1}}\right)\mathcal{A}_n(p_1,\ldots,p_n), \quad (2.7.7)$$

where we have used the following form for the polarization vector of the photon

$$\epsilon_+=\frac{1}{\sqrt{2}}\Big(\bar{\zeta}_{n+1},1,-i,\bar{\zeta}_{n+1}\Big). \quad (2.7.8)$$

A planar zero is now obtained by setting

$$\sum_{i\in\text{out}}\frac{e_i}{\zeta_i-\zeta_{n+1}}=\sum_{j\in\text{in}}\frac{e_j}{\zeta_j-\zeta_{n+1}}, \quad (2.7.9)$$

with $\zeta_1,\ldots,\zeta_{n+1}\in\mathbb{R}$. To compare with previous results, it is convenient to recast (2.7.9) in the reference frame defined by Eqs. (2.1.3) and (2.1.4). Setting $\zeta_1=\infty$ and $\zeta_2=0$,

$$\sum_{i=3}^{n}\frac{e_i}{\zeta_i-\zeta_{n+1}}+\frac{e_2}{\zeta_{n+1}}=0 \quad\Longrightarrow\quad \zeta_{n+1}\sum_{i=3}^{n}e_i\prod_{i\neq\ell=3}^{n}(\zeta_\ell-\zeta_{n+1})+e_2\prod_{\ell=3}^{n}(\zeta_\ell-\zeta_{n+1})=0. \quad (2.7.10)$$

The condition now is expressed in terms of a homogeneous polynomial of degree $n-2$ in the $n-1$ stereographic coordinates $(\zeta_3,\ldots,\zeta_{n+1})$ parametrizing the momenta of the outgoing particles. Particularizing the analysis to the five point amplitude and hard particles with charges $e_1=e_4=e$, $e_2=e_3=e'$, we have

$$e'\zeta_5(\zeta_3-\zeta_5)+e\zeta_5(\zeta_4-\zeta_5)+e'(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)=0. \quad (2.7.11)$$

which is equivalent to (2.3.18) upon setting the projective coordinates defined in (2.3.12).

2.8 Closing remarks

It is indeed surprising that planar zeros of scattering amplitudes in (super) Yang-Mills theories are determined by equations that are invariant under projective transformations of the stereographic coordinates associated with the directions of flight of the outgoing gauge bosons. In this chapter we have shown that this is not a generic feature of field theories: while scalar fields coupled to gauge bosons preserve the projective nature of planar zeros, pure scalar theories have planar zeros that are not determined by projective curves. We have checked this explicitly in the case of the five-point amplitude in a theory of biadjoint scalars with cubic interactions.

The projective nature of gauge planar zeros is also fragile with respect to the inclusion of string effects. We have seen how the α' corrections to the five gluon amplitude introduces terms which do not share the projective structure of the field theory result.

The features of planar gravitational scattering differ in many aspects from those of gauge theories. Due to the peculiar features of three-dimensional gravity, odd-multiplicity amplitudes are zero in the planar limit while for even multiplicities they are only nonzero when helicity is conserved. We have checked this fact explicitly in various cases. String corrections to the field theory amplitude are generically nonvanishing in the planar limit, independently of their helicities and multiplicities, thus correcting the strong constraints imposed by the results of [39].

There are some intriguing elements in the interplay between planar zeros and soft limits in gauge theories that are worth exploring. Although planar zeros are expected to be corrected by quantum effects, the very fact that they are determined by the soft limit indicate that they might be of relevance for the infrared properties of the theory. In particular, it would be interesting to explore whether planar zeros are of any relevance for the asymptotic symmetries for theories like QED [38, 58–61].

Chapter 3

Sudakov Representation of CHY Scattering Equations

3.1 Introduction

Over the last two decades, large progress has been made in the field of mathematical physics, seeking for an efficient way of computing scattering amplitudes [4–8], in contrast to the sometime laborious traditional methods relying on Feynman diagramatics. Among them, the Cachazo-He-Yuan (CHY) formalism [62–64] has found its own place as one of the most versatile. Its construction leads to a direct relation between elements in Field and String theory; and more importantly, this representation of general scattering amplitudes presents many advantages that will be reviewed in Chapter 4. The pertinent structure for the present chapter though, comes from a single but insightful object on which the whole formalism is based: the *scattering equations* (SE)

$$\mathcal{S}_i(\sigma) \equiv \sum_{j\neq i}^{n} \frac{s_{ij}}{\sigma_i - \sigma_j} = 0 \qquad \text{for} \qquad i = 1, \ldots, n\ . \tag{3.1.1}$$

In general terms, these equations provide a map between the space of Mandelstam invariants $s_{ij} = (p_i + p_j)^2$ describing a whole set of n null D-dimensional vectors constrained by momentum conservation and the moduli space of spheres with n-marked points σ_i. We will see in the next section that the actual strength of this map lies in that it naturally introduces all the pole structure of scattering amplitudes arising from locality and unitarity right into the definition of the space of punctured spheres, thus giving rise to a creative and compact amplitude representation in which all the singularities are traced-out. The full tree-level amplitude is later recovered as an integral formula evaluated on the support of the SE.

Having said that, the main drawback of the new approach appears when trying to obtain the solutions to the SE. After many efforts during the recent years, it is likely that finding all of them analytically for an arbitrary multiplicity n is not an easy task. Some attempts, without much success, can be found in [65, 66] in the general case; and in [67, 68], where the computation simplifies drastically just by working in some specific kinematic regimes. As a consequence, almost the only practical strategy nowadays to take advantage of the formalism is to tackle it as a numerical problem. Different methods such as Monte-Carlo algorithms or bootstrapping techniques can be found in [69, 70] for arbitrary kinematics.

The SE have previously appeared in the literature a few times in the context of String Theory [71–74]; nevertheless, the first systematic appearance inside the formalism with this same formulation of Eq. (3.1.1) was in [75]. Understood in detail at tree-level, there also exist some extensions of the SE at loop-level [76–79] by considering a higher genus space of punctured surfaces. Some interesting structures arise as well —studied under the name of *Ambitwistor String Theory* [80]— when particularizing the SE formalism to 4 dimensions and combining it with the power of the spinor-helicity representation [76, 81].

In this chapter we focus on the physical interpretation of the solutions to the SE in terms of the positions of the associated punctures on the Riemann sphere. We find that Sudakov variables [82], which parametrize outgoing momenta in terms of its projections onto two incoming momenta and a vector transverse to their collision axis, are a very efficient way to present the solutions to the SE, since they naturally encode momentum conservation. When evaluating the scattering amplitudes it is also useful to work in the center-of-mass frame of the two incoming particles. This is equivalent, as we will see, to partially fixing some $SL(2, \mathbb{C})$ redundancy on the sphere, localizing two of the punctures at opposite poles while leaving a third puncture free. The residual symmetry corresponds to the freedom in the choice of the origin for the azimuthal angle with respect to the axis defined by the incoming particles. Choosing this global phase wisely allows for a simple representation of the scattering amplitudes in terms of the position of the punctures on the sphere, which also admit a simple representation themselves. With this in mind, this chapter serves as a first contact for the computations that will be done in Chapter 4, where some of the previous results from Chapters 1 and 2 are retrieved thanks to the new approach. Of course the present chapter also has a relevance by itself showing how Sudakov variables, starting from the study of low-multiplicity amplitudes, seem to be the natural parametrization of the CHY formalism for obtaining the exact solutions to the SE.

In Section 3.2, we give a brief review on the SE formalism. We define the new space of punctured spheres to be dealt with and unveil the physical intuition behind the formulation of the SE as a rational map. Then, in Section 3.3 we discuss in detail the particular solution found in [83], which exists for any number of external particles in four dimensions and which we write in terms of the rapidities and the azimuthal angles of the emitted particles. In Section 3.4, after identifying two of the particles participating in the scattering as incoming, we work in their center-of-mass frame taking the z axis as their direction of flight. This is done through a double scaling limit involving the rapidities and transverse momenta. Section 3.5 is devoted to describe the use of Sudakov variables in the simple case of four-particle scattering. The punctures associated with the outgoing momenta are characterized by a single Sudakov variable and one azimuthal angle, which parametrizes circles on the Riemann sphere. In Section 3.6, we analyze the more complicated case of a five-point collision. Sudakov parametrization is shown to be the best way to analytically solve the SE, whose solutions turn out to be rather cumbersome in terms of Mandelstam invariants. In this case four Sudakov variables and two azimuthal angles are needed to characterize the system of equations and the punctures positions. We show how to obtain a second solution to the SE as the complex conjugate of the one previously discussed. In addition, we also explore the possibility of computing the solutions by taking advantage of one notorious property not discussed in the previous sections: *KLT orthogonality.* Being the first case in which there are more than two solutions, we study the 6-point SE in Section 3.7. Again, Sudakov variables seem to simplify the problem, although this time they only grant information about the modulus of the corresponding punctures. The result is somehow a step forward in favor of the parametrization.

3.2 Momentum space and the punctured sphere

In a scattering problem the kinematic information is codified in a set of n on-shell momenta p_i^μ satisfying energy-momentum conservation

$$\Phi_n = \left\{ (p_1, \ldots, p_n) \in (\mathbb{C}M)^n \middle| \sum_{i=1}^{n} p_i = 0, p_j^2 = 0 \quad \forall j \right\} / SO(1,3) \, , \tag{3.2.1}$$

modulo Lorentz transformations. This is the traditional strategy in describing the kinematics of any n-particle collision. However the CHY framework —and this is the key point where the success of the formalism resides— makes use of a different space, which implicitly encodes more physical information than the standard Φ_n. The moduli space of Riemann spheres with n marked points is defined as

$$\mathcal{M}_{0,n} = \left\{ \sigma \in (\mathbb{CP}^1)^n \middle| \sigma_i \neq \sigma_j \quad \forall i,j \right\} / SL(2,\mathbb{C}) \, , \tag{3.2.2}$$

and is $(n-3)$-dimensional due to the presence of the $SL(2,\mathbb{C})$ symmetry. It will be important later on to notice that, by construction, the singular points of this space correspond to n-punctured spheres in which two of the punctures are very close together; whereas the original one presents no apparent singularities. Despite this fact, the general structure is similar. By direct comparison, one can see that the $SL(2,\mathbb{C})$ group is playing the role of Lorentz trasformations[1]. Its action over the elements of $\mathcal{M}_{0,n}$ can be written as

$$\sigma_i \quad \mapsto \quad \sigma_i' = g \cdot \sigma_i := \frac{A\sigma_i + B}{C\sigma_i + D} \quad \text{with} \quad g \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{C}) \, , \tag{3.2.3}$$

where the double covering of the Lorentz group becomes manifest. More clearly, both elements $g \in SL(2,\mathbb{C})$ and $(-g) \in SL(2,\mathbb{C})$ lead to the same transformation, meaning that the rigurous isomorphism would be $SO(1,3) \cong PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\mathbb{Z}_2$. We will keep the above notation for simplicity.

The relevant structure of the formalism though, is the way in which both spaces Φ_n and $\mathcal{M}_{0,n}$ are identified. The mapping between them is performed through the following

[1]Indeed, there are many other situations in which the $SL(2,\mathbb{C})$ representation of the Lorentz group becomes manifest e.g. spinors transform under this representation.

integral expression

$$p_j^\mu = \oint_{|z-\sigma_j|=\epsilon} \frac{\mathrm{d}z}{2\pi i} \frac{v^\mu(z)}{\prod_{k=1}^n (z-\sigma_k)} \ , \tag{3.2.4}$$

where

$$v^\mu(z) = \sum_{j=1}^n p_j^\mu \prod_{\substack{k=1 \\ k\neq j}}^n (z-\sigma_k) \ . \tag{3.2.5}$$

Note that it is unambiguously well-defined for any dimension D. Nevertheless, getting a precise physical intuition out of it is not straightforward. One solution stems from reformulating and writting it in a rational form by looking in detail at the properties of the function $v^\mu(z)$. Due to momentum conservation, it is a vector-valued polynomial of degree $n-2$ satisfying $v(z)^2 = 0$. According to Cauchy's integral formula in Eq. (3.2.4), it is clear that $v(\sigma_i)^2 = p_i^2 = 0$ for every puncture on the sphere. Then, applying the derivative with respect to z, it is also true that $v(z) \cdot v'(z) = 0$. In particular, evaluated on the punctures it is easy to see that

$$v(\sigma_i) \cdot v'(\sigma_i) = \frac{\prod_{k\neq i}(\sigma_i - \sigma_j)^2}{2} \sum_{j\neq i} \frac{s_{ij}}{\sigma_i - \sigma_j} = 0 \qquad \forall i \ , \tag{3.2.6}$$

where we have introduced again the Mandelstam invariants $s_{ij} = (p_i + p_j)^2 = 2\, p_i \cdot p_j$.

Hence, we have ended up with a reformulation of the map in Eq. (3.2.4) as a system of equations

$$\mathcal{S}_i(\sigma) \equiv \sum_{j\neq i}^n \frac{s_{ij}}{\sigma_i - \sigma_j} = 0 \qquad \forall i \ . \tag{3.2.7}$$

They are given the name of *scattering equations* (SE) and become the main ingredient for the computation of scattering amplitudes in the CHY formalism as we will see in Chapter 4. The system of equations can also be rearranged into a polynomial form [see Eq. (3.7.19)].

It is important to realize that the SE are $SL(2,\mathbb{C})$ invariant —i.e. $\mathcal{S}(\sigma_i) = \mathcal{S}(\sigma_i') = 0$— . As a consequence, although there are n equations, only $n-3$ are linearly independent due to the identities

$$\sum_{i=1}^n \mathcal{S}_i(z) = 0 \ , \qquad \sum_{i=1}^n z_i \mathcal{S}_i(z) = 0 \ , \qquad \sum_{i=1}^n z_i^2 \mathcal{S}_i(z) = 0 \ , \tag{3.2.8}$$

leading to a total of $(n-3)!$ inequivalent[2] solutions mapping the space of kinematic invariants into the moduli space of n-punctured spheres.

The SE first appeared in Ref. [74] in the study of the ground state configuration for

[2] Not related by a $SL(2,\mathbb{C})$ transformation.

the Koba-Nielsen representation of scattering amplitudes of open strings,

$$A_n = \int \mathrm{d}\sigma_2 \ldots \mathrm{d}\sigma_{n-2} \prod_{\substack{i,j=1 \\ i>j}}^{n-1} \sigma_{ij}^{-2\alpha' p_i \cdot p_j}, \tag{3.2.9}$$

where $0 = \sigma_1 < \sigma_2 < \cdots < \sigma_{n-1} = 1$. The dominant saddle-point region was investigated by Gross and Mende [71, 72] in the closed string case and by Gross and Mañes for open strings [73]. In both cases, all s_{ij} are taken to be large simultaneously, corresponding precisely to Eq. (3.2.7).

As it was mentioned before, the SE in Eq. (3.2.7) have an easier physical interpretation on the kinematics of an n-particle configuration. Let us think about a generic scattering process of four particles for simplicity. We know that the corresponding scattering amplitude is going to have three physically meaninful singularities —factorization channels— located at $s_{12} \to 0$, $s_{13} \to 0$ and $s_{14} \to 0$. In parallel, we know that the space $\mathcal{M}_{0,4}$ has three singularities as one of the punctures moves closer to the remaining three. Wouldn't it be ideal that the SE map the poles from the space of kinematic invariants to singularities of $\mathcal{M}_{0,4}$?

$$\begin{pmatrix} s_{12} \to 0 \\ s_{13} \to 0 \\ s_{14} \to 0 \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} \sigma_{12} \to 0 \\ \sigma_{13} \to 0 \\ \sigma_{14} \to 0 \end{pmatrix} . \tag{3.2.10}$$

Imagine that we fix the $SL(2,\mathbb{C})$ redundancy by setting arbitrarily[3] three of the punctures in the Riemann sphere to $\sigma_2 = 0$, $\sigma_3 = 1$ and $\sigma_4 = \infty$. The simplest function for the remaining puncture σ_1 that performs the exact identification in Eq. (3.2.10) is

$$\sigma_1 = \frac{-s_{12}}{s_{14}} \qquad \Rightarrow \qquad \begin{cases} \sigma_1 \to 0 = \sigma_2 & \text{for} \quad s_{12} \to 0 \,, \\ \sigma_1 \to 1 = \sigma_3 & \text{for} \quad s_{13} \to 0 \,, \\ \sigma_1 \to \infty = \sigma_4 & \text{for} \quad s_{14} \to 0 \,, \end{cases} \tag{3.2.11}$$

in agreement with the $SL(2,\mathbb{C})$ fixing. Massaging a little this expression and taking advantage of the fixing, it is straightforwad to see that

$$\begin{aligned} \sigma_1 = \frac{-s_{12}}{s_{14}} = \frac{-s_{12}}{s_{32}} \quad &\Rightarrow \quad \frac{\sigma_1 - \sigma_2}{\sigma_3 - \sigma_2} = \frac{-s_{12}}{s_{32}} \\ &\Rightarrow \quad \frac{s_{32}}{\sigma_3 - \sigma_2} = \frac{-s_{12}}{\sigma_1 - \sigma_2} \\ &\Rightarrow \quad \frac{s_{12}}{\sigma_1 - \sigma_2} + \frac{s_{32}}{\sigma_3 - \sigma_2} = 0 \\ &\Rightarrow \quad \frac{s_{12}}{\sigma_1 - \sigma_2} + \frac{s_{32}}{\sigma_3 - \sigma_2} + \frac{s_{42}}{\sigma_4 - \sigma_2} = 0 \,, \end{aligned} \tag{3.2.12}$$

[3] We choose this particular fixing for the sake of simplicity, although the derivation is completely $SL(2,\mathbb{C})$ invariant.

which turns out to be precisely one of the SE: $\mathcal{S}_2(\sigma) = 0$. The same result applies to arbitrary multiplicity n. Remarkably, by a simple computation, we have checked that the SE are not just some random mapping, but implicitly insert information about locality and unitarity of scattering amplitudes directly into the very definition of the space $\mathcal{M}_{0,n}$.

3.3 Fairlie's solution to the scattering equations

In this section we discuss a definite solution to the SE discussed by Fairlie in [83] (see also [74]), which always exists for any multiplicity n. It has the form

$$\begin{aligned}\sigma_j &= \frac{p_j^0 + p_j^3}{p_j^1 - ip_j^2} = \frac{\left(p_j^0 + p_j^3\right)\left(p_j^1 + ip_j^2\right)}{(p_j^1)^2 + (p_j^2)^2} \\ &= \frac{\left(p_j^0 + p_j^3\right)\left(p_j^1 + ip_j^2\right)}{(p_j^0)^2 - (p_j^3)^2} = \frac{p_j^1 + ip_j^2}{p_j^0 - p_j^3},\end{aligned} \tag{3.3.1}$$

where we work in $D = 4$ with the mostly-minus signature. Since σ_i admits two expressions in terms of the momentum components, we can write two alternative identities to be satisfied by the differences σ_{ij}

$$\begin{aligned}\sigma_{ij}\left(p_i^1 - ip_i^2\right)\left(p_j^0 - p_j^3\right) &= p_i \cdot p_j - p_i^0 p_j^3 + p_i^3 p_j^0 - ip_i^1 p_j^2 + ip_i^2 p_j^1, \\ \sigma_{ij}\left(p_j^1 - ip_j^2\right)\left(p_i^0 - p_i^3\right) &= -p_i \cdot p_j + p_j^0 p_i^3 - p_j^3 p_i^0 + ip_j^1 p_i^2 - ip_j^2 p_i^1.\end{aligned} \tag{3.3.2}$$

Subtracting both equations, we arrive at the expression

$$\left(p_i^1 - ip_i^2\right)\left(p_j^0 - p_j^3\right) - \left(p_j^1 - ip_j^2\right)\left(p_i^0 - p_i^3\right) = 2\frac{p_i \cdot p_j}{\sigma_{ij}}. \tag{3.3.3}$$

We can use this identity to explicitly check that (3.3.1) is indeed a solution to the SE. Summing over j with $j \neq i$ we have

$$\begin{aligned}2\sum_{\substack{j=1\\j\neq i}}^{n} \frac{p_i \cdot p_j}{\sigma_{ij}} &= \left(p_i^1 - ip_i^2\right)\sum_{\substack{j=1\\j\neq i}}^{n}\left(p_j^0 - p_j^3\right) - \left(p_i^0 - p_i^3\right)\sum_{\substack{j=1\\j\neq i}}^{n}\left(p_j^1 - ip_j^2\right) \\ &= -\left(p_i^1 - ip_i^2\right)\left(p_i^0 - p_i^3\right) + \left(p_i^0 - p_i^3\right)\left(p_i^1 - ip_i^2\right) = 0,\end{aligned} \tag{3.3.4}$$

where we have made use of momentum conservation.

It is possible to bring these solutions into a more physical representation if we use the following parametrization of on-shell momenta p_j

$$p_j = p_j^{\perp}\left(\cosh Y_j, \cos\phi_j, \sin\phi_j, \sinh Y_j\right), \tag{3.3.5}$$

where Y_j is the rapidity, ϕ_j the azimuthal angle, and the overall scale $p_j^{\perp}$ equals the modulus of the transverse component of the momentum. To connect this representation with the one in terms of the n-punctured sphere, we notice that p_j can be alternatively written as

$$p_j = \omega_j \left(1, \mathbf{u}_j\right), \tag{3.3.6}$$

where we have introduced the unit vector

$$\mathbf{u}_j = (x_j, y_j, z_j), \qquad \mathbf{u}_j^2 = 1, \tag{3.3.7}$$

and ω_j is the energy of the j-th particle. Using this parametrization it is glaring how a null momentum is completely specified by the energy of the particle and its direction of flight, corresponding to a point on $\mathbb{R} \times \mathbb{S}^2$. Points on the celestial sphere $\mathbb{S}^2$ can be parametrized either using stereographic coordinates ζ_j or the polar and azimuthal angles (θ_j, ϕ_j). They are related by the following identities

$$\begin{aligned}
x_j &= \sin\theta_j \cos\phi_j = \frac{2e^{Y_j}\cos\phi_j}{1+e^{2Y_j}} = \frac{\zeta_j + \bar{\zeta}_j}{1+\zeta_j\bar{\zeta}_j}, \\
y_j &= \sin\theta_j \sin\phi_j = \frac{2e^{Y_j}\sin\phi_j}{1+e^{2Y_j}} = i\frac{\bar{\zeta}_j - \zeta_j}{1+\zeta_j\bar{\zeta}_j}, \\
z_j &= \cos\theta_j = \frac{e^{2Y_j}-1}{1+e^{2Y_j}} = \frac{\zeta_j\bar{\zeta}_j - 1}{1+\zeta_j\bar{\zeta}_j},
\end{aligned} \tag{3.3.8}$$

which can be inverted to give

$$\begin{aligned}
\zeta_j = e^{Y_j}e^{i\phi_j} &= \frac{\sin\theta_j}{1-\cos\theta_j}e^{i\phi_j} = \cot\frac{\theta_j}{2}e^{i\phi_j}, \\
\bar{\zeta}_j = e^{Y_j}e^{-i\phi_j} &= \frac{\sin\theta_j}{1-\cos\theta_j}e^{-i\phi_j} = \cot\frac{\theta_j}{2}e^{-i\phi_j}.
\end{aligned} \tag{3.3.9}$$

This leads to the following parametrization of the particle momenta in terms of its energy and the stereographic coordinates on $\mathbb{S}^2$

$$p_j = \omega_j \left(1, \frac{\zeta_j + \bar{\zeta}_j}{1+\zeta_j\bar{\zeta}_j}, i\frac{\bar{\zeta}_j - \zeta_j}{1+\zeta_j\bar{\zeta}_j}, \frac{\zeta_j\bar{\zeta}_j - 1}{1+\zeta_j\bar{\zeta}_j}\right). \tag{3.3.10}$$

Using the previous representation of the particle momenta, we see that Fairlie's solution (3.3.1) to the SE is simply given by

$$\sigma_j = \zeta_j = e^{Y_j + i\phi_j}. \tag{3.3.11}$$

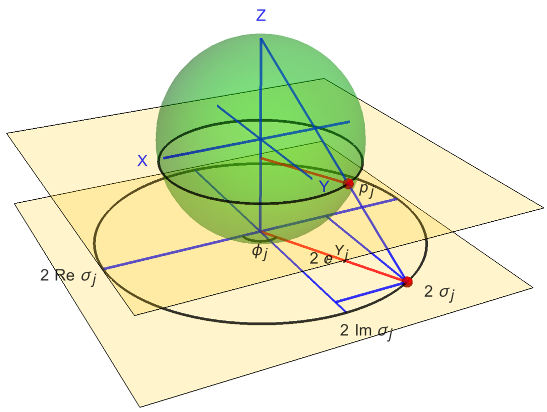


Figure 3.1: Geometric interpretation of the rapidity Y_j and azimuthal angle ϕ_j.

Since we will make frequent use of this representation in the following, some remarks are in order. In Fig. 3.1 we have represented a point in the celestial sphere and its image on the complex plane whose origin coincides with the south pole. The direction of flight of a particle with momentum p_j labelled by the complex coordinate ζ_j is mapped onto the point $2\sigma_j$ on that plane. At fixed rapidity Y_j, the points lie on a circumference of radius $2e^{Y_j}$ parametrized by the azimuthal angle ϕ_j.

3.4 Incoming momenta

In this section we investigate the structure on the punctured sphere for the two incoming particles with momenta p and q in a general process in which the particles in the final state have momenta p_i (with $i = 1, \ldots, n-2$). We will consider the case when the two incoming particles' spatial momenta lie along the z axis. It is convenient to work first with the parametrization in terms of rapidities and azimuthal angles introduced in Eq. (3.3.5)

$$\begin{aligned} p &= \ell \left(\cosh Y_p, \cos\phi, \sin\phi, \sinh Y_p\right), \\ q &= \ell \left(\cosh Y_q, -\cos\phi, -\sin\phi, \sinh Y_q\right), \end{aligned} \tag{3.4.1}$$

where we have set both transverse momenta equal, $p^\perp = q^\perp \equiv \ell$. To study the limit of vanishing transverse momenta, we take $\ell \to 0$ and $|Y_p|, |Y_q| \to \infty$, while keeping the center-of-mass energy

$$s = 2p\cdot q = 2\ell^2 \Big[1 + \cosh\left(Y_p - Y_q\right)\Big] \tag{3.4.2}$$

finite. This limit can be implemented by introducing a parameter ϵ

$$Y_p = -Y_q = -\log\epsilon, \tag{3.4.3}$$

that we eventually take to zero. A look at Eq. (3.4.2) shows that in order to keep s finite we are forced to take the double scaling limit

$$\epsilon \longrightarrow 0, \qquad \ell \longrightarrow 0 \quad \text{with} \quad \frac{\ell}{\epsilon} = \sqrt{s}, \tag{3.4.4}$$

in which the incoming momenta take the form

$$\begin{aligned} p &\longrightarrow \frac{\sqrt{s}}{2}(1,0,0,1), \\ q &\longrightarrow \frac{\sqrt{s}}{2}(1,0,0,-1). \end{aligned} \tag{3.4.5}$$

We can rephrase this double scaling in terms of the position of the corresponding punctures on the sphere $\{\sigma_p, \sigma_q\}$, which satisfy the identity

$$\frac{\sigma_p}{\sigma_q} + \frac{\sigma_q}{\sigma_p} = 2 - \frac{s}{\ell^2}. \tag{3.4.6}$$

Eq. (3.3.11) shows that for small ϵ the two punctures are located on a small circle around the north and south poles of the Riemann sphere, which shrinks to a point when $\epsilon \to 0$, namely

$$\begin{aligned} \sigma_p &= e^{Y_p + i\phi} = \frac{e^{i\phi}}{\epsilon} \longrightarrow \infty, \\ \sigma_q &= -e^{Y_q + i\phi} = -\epsilon\, e^{i\phi} \longrightarrow 0. \end{aligned} \tag{3.4.7}$$

These punctures can be alternatively labelled by the unit vectors $\mathbf{u}_p$ and $\mathbf{u}_q$ defined by Eq. (3.3.6). In our case, they take the form

$$\begin{aligned} \mathbf{u}_p &= \left(\frac{\cos\phi}{\cosh Y_p}, \frac{\sin\phi}{\cosh Y_p}, \tanh Y_p\right), \\ \mathbf{u}_q &= \left(-\frac{\cos\phi}{\cosh Y_q}, -\frac{\sin\phi}{\cosh Y_q}, \tanh Y_q\right), \end{aligned} \tag{3.4.8}$$

whose projections onto the equatorial plane lie on circles with respective radii

$$\begin{aligned} R_p &= \frac{1}{\cosh Y_p}, \\ R_q &= \frac{1}{\cosh Y_q}, \end{aligned} \tag{3.4.9}$$

which shrink to zero as $|Y_{p,q}| \to \infty$ (i.e., $\epsilon \to 0$).

The geometric setup for the configuration discussed here is illustrated in Fig. 3.2, where

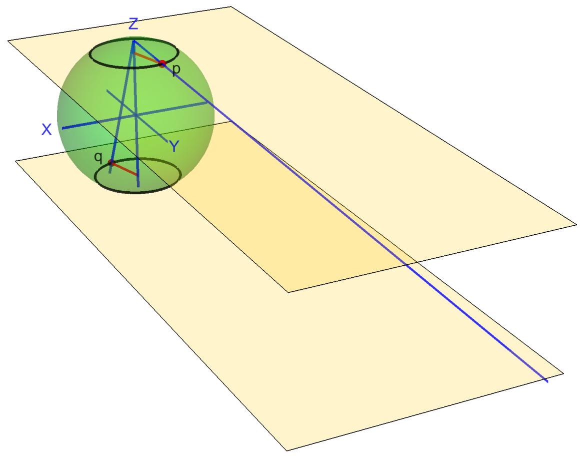


Figure 3.2: Stereographic projection for two incoming particles whose momenta lie close to the z axis.

we show the punctures associated with the incoming particles very close to the north and south poles of the Riemann sphere. The value of ϕ is ambiguous for points on the z axis and without loss of generality we can set it to zero from now on, since this angle is a mere artefact of the way we take the limit. On the complex plane this means that the limits $\sigma_q \to 0$ and $\sigma_p \to \infty$ are taken along the real axis.

3.5 The four-point case

After introducing our setup and conventions, we turn to study the formulation of the SE formalism in terms of Sudakov parameters. We begin with the simplest case, that of a general four-point scattering amplitude with incoming and outgoing momenta respectively given by p, q and p', q', which are constrained by momentum conservation

$$p + q - p' - q' = 0. \tag{3.5.1}$$

We parametrize the two incoming momenta p and q as explained in the previous Section.

3.5.1 Punctures on the Riemann sphere

In the CHY formalism [63,64], the momenta $\{p, q, p', q'\}$ are mapped into the moduli space of spheres with four punctures, located respectively at the points $\{\sigma_p, \sigma_q, \sigma_{p'}, \sigma_{q'}\} \in \mathbb{CP}^1$. This is implemented by the identities

$$p^\mu = \oint\limits_{|z-\sigma_p|=\epsilon} \frac{\mathrm{d}z}{2\pi i} \omega^\mu(z),$$

$$
\begin{aligned}
q^\mu &= \oint\limits_{|z-\sigma_q|=\epsilon} \frac{\mathrm{d}z}{2\pi i}\omega^\mu(z), \\
p'^\mu &= -\oint\limits_{|z-\sigma_{p'}|=\epsilon} \frac{\mathrm{d}z}{2\pi i}\,\omega^\mu(z), \\
q'^\mu &= -\oint\limits_{|z-\sigma_{q'}|=\epsilon} \frac{\mathrm{d}z}{2\pi i}\,\omega^\mu(z),
\end{aligned}
\tag{3.5.2}
$$

where the meromorphic function $\omega^\mu(z)$ is fully determined by the condition that it has poles at the location of the punctures whose residues are the corresponding particle momenta

$$
\omega^\mu(z) = \frac{p^\mu}{z-\sigma_p} + \frac{q^\mu}{z-\sigma_q} - \frac{p'^\mu}{z-\sigma_{p'}} - \frac{q'^\mu}{z-\sigma_{q'}}. \tag{3.5.3}
$$

The incoming momenta are parametrized as shown in Eq. (3.4.1) with $\phi = 0$. For the outgoing particles, on the other hand, we write their momenta introducing a Sudakov [82] representation. Due to momentum conservation (3.5.1), it is enough to parametrize the combination

$$
q_1 \equiv p - p' = \alpha\, p + \beta\, q + \mathbf{q}_1, \tag{3.5.4}
$$

with

$$
\mathbf{q}_1 = q_1^\perp \,(0, \cos\theta_1, \sin\theta_1, 0)\,. \tag{3.5.5}
$$

Then, the momentum p' can be written as

$$
\begin{aligned}
p' &= p - q_1 \\
&= \ell\Big((1-\alpha)\cosh Y_p - \beta\cosh Y_q, 0, 0, (1-\alpha)\sinh Y_p - \beta\sinh Y_q\Big) \\
&+ \Big(0, (1-\alpha+\beta)\,\ell - q_1^\perp\cos\theta_1, -q_1^\perp\sin\theta_1, 0\Big), \\
&\longrightarrow \left(\frac{\sqrt{s}}{2}\,(1-\alpha-\beta), -q_1^\perp\cos\theta_1, -q_1^\perp\sin\theta_1, \frac{\sqrt{s}}{2}(1-\alpha+\beta)\right),
\end{aligned}
\tag{3.5.6}
$$

where in the last expression we have taken the double scaling limit (3.4.4). From this we read the particle energy

$$
\omega_{p'} = \frac{\sqrt{s}}{2}\,(1-\alpha-\beta)\,, \tag{3.5.7}
$$

whereas the on-shell condition leads to

$$0 = p'^2 = -s(1-\alpha)\beta - (q_1^\perp)^2 \qquad \Longrightarrow \qquad |Q_1|^2 \equiv (q_1^\perp)^2 = s(\alpha-1)\beta, \tag{3.5.8}$$

where we have introduced the notation

$$Q_j = q_j^\perp e^{i\theta_j}. \tag{3.5.9}$$

We repeat the same calculation for the momentum q' of the second outgoing particle. In terms of the Sudakov parameters, it reads

$$\begin{aligned} q' &= q + q_1 \\ &= \ell\Big(\alpha\cosh Y_p + (1+\beta)\cosh Y_q, 0, 0, \alpha\sinh Y_p + (1+\beta)\sinh Y_q\Big) \\ &+ \Big(0, (\alpha-1-\beta)\,\ell + q_1^\perp\cos\theta_1, q_1^\perp\sin\theta_1, 0\Big) \\ &\longrightarrow \left(\frac{\sqrt{s}}{2}\,(1+\alpha+\beta), q_1^\perp\cos\theta_1, q_1^\perp\sin\theta_1, \frac{\sqrt{s}}{2}(-1+\alpha-\beta)\right), \end{aligned} \tag{3.5.10}$$

where we have reabsorbed a sign in a shift of θ_1 by π. Comparing with the expression for p' in Eq. (3.5.6) we see that this reflects the fact that, in the center-of-mass frame, the two outgoing particles fly in opposite directions and therefore their azimuthal angles differ by π. The energy of the particle is given by

$$\omega_{q'} = \frac{\sqrt{s}}{2}\,(1+\alpha+\beta)\,, \tag{3.5.11}$$

whereas the on-shell condition $q'^2 = 0$ leads to the constraint

$$0 = q'^2 = s\alpha(1+\beta) - (q_1^\perp)^2 \qquad \Longrightarrow \qquad |Q_1|^2 \equiv (q_1^\perp)^2 = s\alpha(1+\beta). \tag{3.5.12}$$

Consistency with the value of $|Q_1|^2$ found from the on-shell condition $p'^2 = 0$ in Eq. (3.5.8) implies that α and β are not independent, but rather satisfy

$$\alpha + \beta = 0. \tag{3.5.13}$$

This condition implies that

$$\omega_{p'} = \omega_{q'} = \frac{\sqrt{s}}{2}, \tag{3.5.14}$$

as it behoves a four particle scattering in the center-of-mass frame.

Let us recall that for the four-point function, the SE only have one solution. Thus, it is enough to consider Fairlie's solution (3.3.1) reviewed in Section 3.3. This being the case, the complex coordinate of the puncture in the sphere associated with the momentum p' is given by

$$\sigma_{p'} \equiv e^{Y_{p'}+i\phi_{p'}} = \frac{Q_1}{\beta\sqrt{s}} = \sqrt{\frac{1-\alpha}{\alpha}}e^{i(\theta_1+\pi)}, \tag{3.5.15}$$

where in using (3.5.8) to write the result in terms of Q_1 we have made a choice of phase for the square root. In addition, the projection of the associated unit vector $\mathbf{u}_{p'}$

$$\mathbf{u}_{p'} = \frac{2}{\sqrt{s}}\Big(q_1^\perp\cos(\theta_1+\pi), q_1^\perp\sin(\theta_1+\pi), \frac{\sqrt{s}}{2}(1-2\alpha)\Big), \tag{3.5.16}$$

onto the equatorial plane lies on a circumference with radius

$$R_{p'} = 2\sqrt{\alpha(1-\alpha)}, \tag{3.5.17}$$

where we have used the on-shell condition (3.5.8). Going to the Riemann sphere representation, the complex coordinate of the puncture associated with the particle of momentum q' is

$$\sigma_{q'} \equiv e^{Y_{q'}+i\phi_{q'}} = \frac{Q_1}{(1-\alpha)\sqrt{s}} = \sqrt{\frac{\alpha}{1-\alpha}}e^{i\theta_1}, \tag{3.5.18}$$

where our choice of phase is consistent with the one used for $\sigma_{p'}$ in Eq. (3.5.15). Thus, we conclude

$$\sigma_{q'} = -\frac{1}{\sigma^*_{p'}} = \sqrt{\frac{\alpha}{1-\alpha}}e^{i\theta_1}, \tag{3.5.19}$$

indicating that the two punctures are located on antipodal points on the sphere. This becomes obvious when computing the components of the unit vector $\mathbf{u}_{q'}$

$$\mathbf{u}_{q'} = \frac{2}{\sqrt{s}}\left(q_1^\perp\cos\theta_1, q_1^\perp\sin\theta_1, -\frac{\sqrt{s}}{2}(1-2\alpha)\right). \tag{3.5.20}$$

Now, since after imposing (3.5.13) we see that $\mathbf{u}_{q'} = -\mathbf{u}_{p'}$, the projection of both vectors on the equatorial plane defines the same loci, namely a circumference with radius [cf. Eq. (3.5.17)]

$$R_{p'} = R_{q'} = 2\sqrt{\alpha(1-\alpha)}, \tag{3.5.21}$$

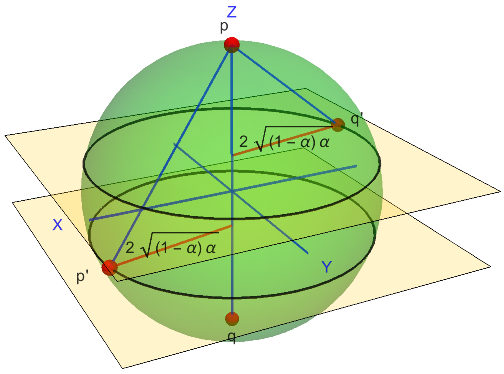


Figure 3.3: Punctures on the Riemann sphere for the four particle scattering with momenta $p+q \longrightarrow p'+q'$. In the limits $\alpha \to 0, 1$ the outgoing punctures collide with the incoming ones located at the poles.

whereas their components along the direction of the incoming particles are

$$Z_{p'} = -Z_{q'} = 1 - 2\alpha. \tag{3.5.22}$$

In Fig. 3.3 we provide a pictorial example of the parametrization proposed above. The boundary of the moduli space of the sphere with four punctures is approached in the limits $\alpha \to 1$ or $\alpha \to 0$. They correspond to the coincidence limit in which the punctures associated with the outgoing particles collide with those of the incoming ones, located at the north and south pole of the Riemann sphere. In the case $\alpha = \frac{1}{2}$ the radii $R_{p'} = R_{q'} = 2\sqrt{\alpha(1-\alpha)}$ reach the maximum value and the outgoing particles are emitted along the equatorial plane.

3.5.2 Scattering Equations

In order to write the SE using the Sudakov parametrization, we need to compute the Mandelstam invariants where, according to our conventions $p_1 = p$, $p_2 = q$, $p_3 = -p'$, and $p_4 = -q'$. For four particle scattering, they have the following explicit form

$$s_{pq} = s_{p'q'} = s, \tag{3.5.23}$$
$$s_{pp'} = s_{qq'} = -q_1^2 = -t = s\alpha, \tag{3.5.24}$$
$$s_{pq'} = s_{qp'} = -u = s(1-\alpha). \tag{3.5.25}$$

Since s is the only dimensionful quantity available, we use rescaled variables

$$\begin{aligned} s_{ij} &= s\hat{s}_{ij}, \\ Q_i &= \sqrt{s}\hat{Q}_i. \end{aligned} \tag{3.5.26}$$

It is straightforward to check that the SE associated with p is trivially satisfied

$$\frac{\mathcal{S}_p}{s} = \frac{\hat{s}_{pq}}{\sigma_{pq}} - \frac{\hat{s}_{pp'}}{\sigma_{pp'}} - \frac{\hat{s}_{pq'}}{\sigma_{pq'}} = 0, \tag{3.5.27}$$

since we have $\sigma_p = \infty$. In the case of the SE associated to q

$$\frac{\mathcal{S}_q}{s} = \frac{\hat{s}_{pq}}{\sigma_{qp}} - \frac{\hat{s}_{qq'}}{\sigma_{qq'}} - \frac{\hat{s}_{qp'}}{\sigma_{qp'}}, \tag{3.5.28}$$

we have a nontrivial cancellation. The first term vanishes again because $\sigma_p = \infty$ and we can use the explicit expressions

$$\sigma_{p'} = -\frac{\hat{Q}_1}{\alpha}\,, \qquad\qquad \sigma_{q'} = \frac{\hat{Q}_1}{1-\alpha}\,, \tag{3.5.29}$$

together with $\sigma_q = 0$. Using this Sudakov representation, it is easy to check that the SE is fulfilled

$$\frac{\mathcal{S}_q}{s} = \frac{\hat{s}_{qq'}}{\sigma_{q'}} + \frac{\hat{s}_{qp'}}{\sigma_{p'}} = -\frac{\alpha(1-\alpha)}{\hat{Q}_1} + \frac{(1-\alpha)\alpha}{\hat{Q}_1} = 0, \tag{3.5.30}$$

and similarly for the two remaining SE

$$\begin{aligned} \frac{\mathcal{S}_{p'}}{s} &= \frac{\hat{s}_{p'q}}{\sigma_{p'}} - \frac{\hat{s}_{p'q'}}{\sigma_{p'q'}} = -\frac{\alpha(1-\alpha)}{\hat{Q}_1} + \frac{\alpha(1-\alpha)}{\hat{Q}_1} = 0, \\ \frac{\mathcal{S}_{q'}}{s} &= \frac{\hat{s}_{qq'}}{\sigma_{q'}} - \frac{\hat{s}_{p'q'}}{\sigma_{q'p'}} = -\frac{\alpha(1-\alpha)}{\hat{Q}_1} + \frac{\alpha(1-\alpha)}{\hat{Q}_1} = 0. \end{aligned} \tag{3.5.31}$$

In any case, assuming a situation in which we do not know any solution a priori, the SE are simple enough just to compute it:

$$\frac{s_{qp}}{\sigma_q - \sigma_p} + \frac{s_{qp'}}{\sigma_q - \sigma_{p'}} + \frac{s_{qq'}}{\sigma_q - \sigma_{q'}} = \frac{u}{\sigma_{p'}} + \frac{t}{\sigma_{q'}} = 0 \quad \to \quad \frac{\sigma_{p'}}{\sigma_{q'}} = \frac{-u}{t} = \frac{\alpha - 1}{\alpha}\,, \tag{3.5.32}$$

which is in complete agreement with Fairlie's punctures in Eq. (3.5.29) as expected.

3.6 The five-point case

After the analysis of the four-point case, we turn to the scattering of five particles which enjoys some more interesting features, mainly the existence of a second solution to the SE besides Fairlie's. To fix notation, we will now study a generic five-point scattering amplitude of particles with momenta $p + q \to p' + k + q'$ satisfying the momentum conservation identity

$$p + q - p' - k - q' = 0. \tag{3.6.1}$$

3.6.1 Location of the punctures

The mapping between particle momenta and the puncture positions is provided by the relations listed in Eq. (3.5.2) supplemented with the one for k

$$k^\mu = -\oint_{|z-\sigma_k|=\epsilon} \frac{\mathrm{d}z}{2\pi i} \omega^\mu(z) \,, \tag{3.6.2}$$

where now the meromorphic function $\omega^\mu(z)$ is given by

$$\omega^\mu(z) = \frac{p^\mu}{z-\sigma_p} + \frac{q^\mu}{z-\sigma_q} - \frac{p'^\mu}{z-\sigma_{p'}} - \frac{k^\mu}{z-\sigma_k} - \frac{q'^\mu}{z-\sigma_{q'}}. \tag{3.6.3}$$

To parametrize the momenta, we introduce two pairs of Sudakov parameters $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$ such that

$$\begin{aligned}
q_1 &= p - p' = \alpha_1 p + \beta_1 q + \mathbf{q}_1, \\
q_2 &= q' - q = \alpha_2 p + \beta_2 q + \mathbf{q}_2, \\
k &= q_1 - q_2 = (\alpha_1 - \alpha_2)\, p + (\beta_1 - \beta_2)\, q + \mathbf{q}_1 - \mathbf{q}_2,
\end{aligned} \tag{3.6.4}$$

where the transverse vectors have components

$$\mathbf{q}_i = q_i^\perp \Big(0, \cos\theta_i, \sin\theta_i, 0\Big). \tag{3.6.5}$$

Using again the notation introduced in Eq. (3.5.9), and taking the double scaling limit (3.4.4), we have

$$\begin{aligned}
p' &= p - q_1 \\
&= \ell\Big((1-\alpha_1)\cosh Y_p - \beta_1 \cosh Y_q, 0, 0, (1-\alpha_1)\sinh Y_p - \beta_1 \sinh Y_q\Big) \\
&+ \Big(0, (1-\alpha_1+\beta_1)\ell - q_1^\perp \cos\theta_1, -q_1^\perp \sin\theta_1, 0\Big) \\
&\longrightarrow \left(\frac{\sqrt{s}}{2}\,(1-\alpha_1-\beta_1), -q_1^\perp\cos\theta_1, -q_1^\perp \sin\theta_1, \frac{\sqrt{s}}{2}(1-\alpha_1+\beta_1)\right).
\end{aligned} \tag{3.6.6}$$

A similar analysis can be repeated for the remaining two outgoing particles. In terms of the Sudakov parameters, their momenta take the form

$$
\begin{aligned}
q' &= q + q_2 \\
&= \ell\Big(\alpha_2 \cosh Y_p + (1+\beta_2)\cosh Y_q, 0, 0, \alpha_2 \sinh Y_p + (1+\beta_2)\sinh Y_q\Big) \\
&+ \Big(0, (\alpha_2 - \beta_2 - 1)\ell + q_2^\perp \cos\theta_2, q_2^\perp \sin\theta_2, 0\Big) \\
&\longrightarrow \left(\frac{\sqrt{s}}{2}\,(1+\alpha_2+\beta_2), q_2^\perp \cos\theta_2, q_2^\perp \sin\theta_2, \frac{\sqrt{s}}{2}(-1+\alpha_2-\beta_2)\right), \qquad (3.6.7) \\
k &= q_1 - q_2 \\
&= \ell\bigg((\alpha_1-\alpha_2)\cosh Y_p + (\beta_1-\beta_2)\cosh Y_q, 0, 0, (\alpha_1-\alpha_2)\sinh Y_p + (\beta_1-\beta_2)\sinh Y_q\bigg) \\
&+ \Big(0, q_1^\perp \cos\theta_1 - q_2^\perp \cos\theta_2, q_1^\perp \sin\theta_1 - q_2^\perp \sin\theta_2, 0\Big) \\
&\longrightarrow \Big(\frac{\sqrt{s}}{2}(\alpha_1+\beta_1-\alpha_2-\beta_2), q_1^\perp \cos\theta_1 - q_2^\perp \cos\theta_2, q_1^\perp - q_2^\perp \sin\theta_2, \frac{\sqrt{s}}{2}(\alpha_1-\beta_1-\alpha_2+\beta_2)\Big).
\end{aligned}
$$

The associated energies are read off these expressions to be

$$
\begin{aligned}
\omega_{p'} &= \frac{\sqrt{s}}{2}\,(1-\alpha_1-\beta_1)\,, \\
\omega_{q'} &= \frac{\sqrt{s}}{2}(1+\alpha_2+\beta_2), \qquad (3.6.8) \\
\omega_k &= \frac{\sqrt{s}}{2}(\alpha_1+\beta_1-\alpha_2-\beta_2),
\end{aligned}
$$

which obviously satisfy energy conservation, $\omega_{p'} + \omega_{q'} + \omega_k = \sqrt{s}$. In addition, the on-shell condition for the outgoing momenta fixes the magnitude of the transverse momenta in terms of the Sudakov parameters

$$
\begin{aligned}
p'^2 = 0 \qquad &\Longrightarrow \qquad |Q_1|^2 = s(\alpha_1 - 1)\beta_1, \\
q'^2 = 0 \qquad &\Longrightarrow \qquad |Q_2|^2 = s\alpha_2(1+\beta_2), \qquad (3.6.9) \\
k^2 = 0 \qquad &\Longrightarrow \qquad |Q_1 - Q_2|^2 = s(\alpha_1-\alpha_2)(\beta_1-\beta_2).
\end{aligned}
$$

In fact, combining them we find a further identity

$$
Q_1 Q_2^* + Q_1^* Q_2 = s(\alpha_2 - \beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1)\,. \qquad (3.6.10)
$$

It is important to stress at this point that, unlike the situation encountered in the four-point amplitude, here the on-shell conditions for the outgoing particles do not lead to consistency identities restricting the values of the Sudakov parameters. Thus, whereas in the case of four particles the identity (3.5.13) implies the existence of a single independent Sudakov parameter, in the five-point amplitude the four parameters remain independent.

The coordinates of the punctures associated with each momenta corresponding to Fairlie's solution are given by

$$
\begin{aligned}
\sigma_{p'} &= \frac{Q_1}{\beta_1\sqrt{s}} = \sqrt{\frac{\alpha_1 - 1}{\beta_1}}e^{i\theta_1} = e^{Y_{p'}+i\phi_{p'}}, \\
\sigma_{q'} &= \frac{Q_2}{(1+\beta_2)\sqrt{s}} = \sqrt{\frac{\alpha_2}{1+\beta_2}}e^{i\theta_2} = e^{Y_{q'}+i\phi_{q'}}, \\
\sigma_k &= \frac{Q_1 - Q_2}{(\beta_1-\beta_2)\sqrt{s}} = \frac{\sqrt{(\alpha_1-1)\beta_1}e^{i\theta_1} - \sqrt{(1+\beta_2)\alpha_2}e^{i\theta_2}}{\beta_1 - \beta_2} = e^{Y_k + i\phi_k},
\end{aligned} \tag{3.6.11}
$$

which are the stereographic coordinates labelling the directions of flight of the particles. In order to visualize the position of these punctures, it is convenient to use the unity vectors

$$
\begin{aligned}
\mathbf{u}_{p'} &= \frac{2}{\sqrt{s}(1-\alpha_1-\beta_1)}\Big(-q_1^\perp\cos\theta_1, -q_1^\perp\sin\theta_1, \frac{\sqrt{s}}{2}(1-\alpha_1+\beta_1)\Big), \\
\mathbf{u}_{q'} &= \frac{2}{\sqrt{s}(1+\alpha_2+\beta_2)}\Big(q_2^\perp\cos\theta_2, q_2^\perp\sin\theta_2, \frac{\sqrt{s}}{2}(-1+\alpha_2-\beta_2)\Big), \\
\mathbf{u}_k &= \frac{2}{\sqrt{s}(\alpha_1+\beta_1-\alpha_2-\beta_2)} \\
&\quad\times\Big(q_1^\perp\cos\theta_1 - q_2^\perp\cos\theta_2, q_1^\perp\sin\theta_1 - q_2^\perp\sin\theta_2, \frac{\sqrt{s}}{2}(\alpha_1-\beta_1-\alpha_2+\beta_2)\Big).
\end{aligned} \tag{3.6.12}
$$

Using the expression for $q_i^\perp$ given in Eq. (3.6.9), we see that the projections of $\mathbf{u}_{p'}$ and $\mathbf{u}_{q'}$ lie onto the equatorial plane on circumferences with radii

$$
\begin{aligned}
R_{p'} &= 2\sqrt{\frac{(\alpha_1-1)\beta_1}{(1-\alpha_1-\beta_1)^2}}, \\
R_{q'} &= 2\sqrt{\frac{\alpha_2(1+\beta_2)}{(1+\alpha_2+\beta_2)^2}}.
\end{aligned} \tag{3.6.13}
$$

For the momentum k, we just need to notice that since θ_1 and θ_2 are respectively the arguments of Q_1 and Q_2

$$
\begin{aligned}
q_1^\perp\cos\theta_1 - q_2^\perp\cos\theta_2 &= \mathrm{Re}\Big(Q_1 - Q_2\Big), \\
q_1^\perp\sin\theta_1 - q_2^\perp\sin\theta_2 &= \mathrm{Im}\Big(Q_1 - Q_2\Big).
\end{aligned} \tag{3.6.14}
$$

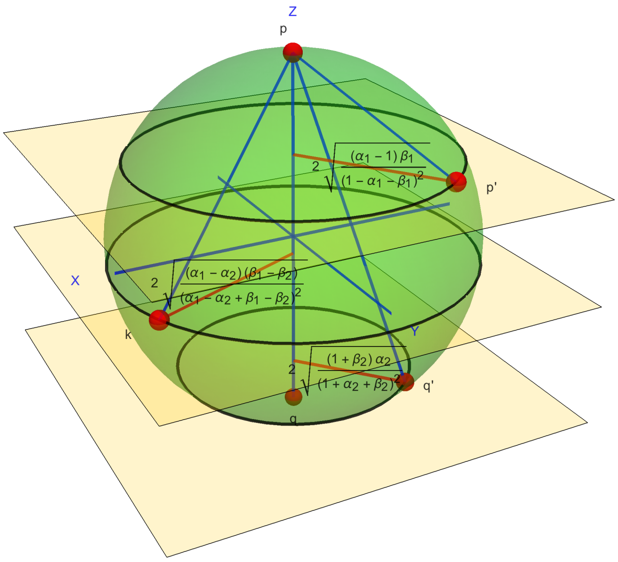


Figure 3.4: Punctures on the Riemann sphere for the five-particle amplitude.

Hence, the equatorial projection of $\mathbf{u}_k$ lies on a circumference of radius

$$R_k = \frac{2|Q_1 - Q_2|}{\alpha_1 + \beta_1 - \alpha_2 - \beta_2} = 2\sqrt{\frac{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)}{(\alpha_1 + \beta_1 - \alpha_2 - \beta_2)^2}}. \tag{3.6.15}$$

In Fig. 3.4 we show a typical configuration for the five punctures on the Riemann sphere. Several factorization channels can be identified in the expressions given in this Section. An interesting one corresponds to $\beta_1, \alpha_2 \to 0$, with both α_1 and $-\beta_2$ not close to 1. This limit sends the puncture associated with p' to the north pole, while the puncture for q' approaches the south pole. In this limit the puncture for k remains at the equator whenever $\alpha_1 + \beta_2 = 0$. Alternatively, we can keep σ_k at the equator by taking $\alpha_1, -\beta_2 \to 1$, with β_1 and α_2 not close to 0. On the other hand, the puncture associated with k moves to the south pole in the limit $\alpha_1 \to \alpha_2$ and to the north pole if $\beta_1 \to \beta_2$.

3.6.2 Scattering Equations

In contrast to what we had in Eq. (3.5.32) for the four-point case, the presence of two independent SE for $n = 5$ with an additional puncture e.g.

$$\begin{aligned} \frac{s_{qp'}}{\sigma_{p'}} + \frac{s_{qq'}}{\sigma_{q'}} + \frac{s_{qk}}{\sigma_k} &= 0 \,, \\ \frac{s_{q'q}}{\sigma_{q'}} + \frac{s_{q'p'}}{\sigma_{q'} - \sigma_{p'}} + \frac{s_{q'k}}{\sigma_{q'} - \sigma_k} &= 0 \,, \end{aligned} \tag{3.6.16}$$

notably complicates any attempt to solve them by brute force. Indeed, written in terms of Mandelstam invariants, the existing two solutions become rather cumbersome

$$\frac{\sigma_{p'}}{\sigma_{q'}} = \frac{\left(s_{p'q'}s_{qk} + s_{p'q'}s_{qq'} + s_{qk}s_{qq'} + \ldots\right) \pm \sqrt{4 s_{qp'} s_{qq'}(\ldots)(\ldots) + (\ldots)^2}}{2 s_{qq'}(s_{qk} + s_{q'k} + s_{qq'})}\,,$$

$$\frac{\sigma_k}{\sigma_{q'}} = \frac{\left(-s_{p'q'}s_{qk} + s_{qp'}s_{q'k} + s_{p'q'}s_{qq'} + \ldots\right) \mp \sqrt{4 s_{qp'} s_{qq'}(\ldots)(\ldots) + (\ldots)^2}}{2 s_{qq'}(s_{p'q'} + s_{qp'} + s_{qq'})}\,. \tag{3.6.17}$$

Let us see how Sudakov parameters allow for significant simplifications. To write the SE, we begin by computing the Mandelstam invariants in terms of the Sudakov variables for the five-point amplitude

$$\begin{aligned} s_{pq} &= s, & s_{p'k} &= -s(\alpha_2 + \beta_2), & s_{q'k} &= s\,(\alpha_1 + \beta_1)\,, \\ s_{pp'} &= -s\beta_1, & s_{qq'} &= s\alpha_2, & s_{pk} &= s(\beta_1 - \beta_2), \\ s_{qk} &= s(\alpha_1 - \alpha_2), & s_{pq'} &= s(1 + \beta_2), & s_{qp'} &= s(1 - \alpha_1), \\ s_{p'q'} &= s(1 - \alpha_1 + \alpha_2 - \beta_1 + \beta_2). \end{aligned} \tag{3.6.18}$$

By inverting these relations, it is possible to express the Sudakov parameters in terms of the invariants as

$$s\,\alpha_1 = s_{q'k} + s_{pp'}, \qquad s\,\alpha_2 = s_{qq'}, \tag{3.6.19}$$

$$s\,\beta_1 = -s_{pp'}, \qquad s\,\beta_2 = -s_{p'k} - s_{qq'}. \tag{3.6.20}$$

We know that for the five-point amplitude there must be two different solutions. One of them is the one found by Fairlie [74,83] that we have expressed in Eq. (3.6.11) in terms of Sudakov parameters. To find the second one, we write the ansatz

$$\begin{aligned} \sigma_{p'} &= C_p\,\hat{Q}_1, \\ \sigma_{q'} &= C_q\,\hat{Q}_2, \end{aligned} \tag{3.6.21}$$

with C_p and C_q two complex constants and we use the rescaled quantities defined in Eq. (3.5.26). A first condition comes from complying with the SE associated to q,

$$\mathcal{S}_q \equiv \frac{1-\alpha_1}{\sigma_{p'}} + \frac{\alpha_2}{\sigma_{q'}} + \frac{\alpha_1 - \alpha_2}{\sigma_k} = 0, \tag{3.6.22}$$

which determines σ_k to be

$$\sigma_k = (\alpha_2 - \alpha_1)\left(\frac{1-\alpha_1}{C_p\hat{Q}_1} + \frac{\alpha_2}{C_q\hat{Q}_2}\right)^{-1}. \tag{3.6.23}$$

Now we impose the SE associated to q', which reads

$$\mathcal{S}_{q'} \equiv \frac{\alpha_2}{\sigma_{q'}} + \frac{1-\alpha_1+\alpha_2-\beta_1+\beta_2}{\sigma_{p'q'}} + \frac{\alpha_1+\beta_1}{\sigma_{kq'}} = 0, \tag{3.6.24}$$

leading to the relation

$$\frac{\sigma_k}{\sigma_{q'}} = \frac{(\alpha_2-\alpha_1)\sigma_{p'}}{\alpha_2\sigma_{p'}+(1-\alpha_1)\sigma_{q'}} = \frac{(\alpha_2-\alpha_1-\beta_1)\sigma_{p'}+(1+\beta_2)\sigma_{q'}}{\alpha_2\sigma_{p'}+(1-\alpha_1-\beta_1+\beta_2)\sigma_{q'}}. \tag{3.6.25}$$

Using the on-shell relations (3.6.9), this equation can be equivalently written as

$$\alpha_2\beta_1\sigma_{p'}^2 - \left(\hat{Q}_1\hat{Q}_2^* + \hat{Q}_1^*\hat{Q}_2\right)\sigma_{p'}\sigma_{q'} + \frac{|\hat{Q}_1|^2|\hat{Q}_2|^2}{\alpha_2\beta_1}\sigma_{q'}^2 = 0. \tag{3.6.26}$$

Assuming $\sigma_{q'} \neq 0$, this is a quadratic equation for the ratio $\frac{\sigma_{p'}}{\sigma_{q'}}$ whose coefficients are expressed only in terms of the Sudakov parameters. Its two solutions are given by

$$\frac{\sigma_{p'}^{(\pm)}}{\sigma_{q'}^{(\pm)}} = \frac{1}{2\alpha_2\beta_1}\left[\hat{Q}_1\hat{Q}_2^* + \hat{Q}_1^*\hat{Q}_2 \pm \sqrt{\left(\hat{Q}_1\hat{Q}_2^* + \hat{Q}_1^*\hat{Q}_2\right)^2 - 4|\hat{Q}_1|^2|\hat{Q}_2|^2}\,\right], \tag{3.6.27}$$

which admits the simpler form

$$\frac{\sigma_{p'}^{(+)}}{\sigma_{q'}^{(+)}} = \frac{\hat{Q}_1\hat{Q}_2^*}{\alpha_2\beta_1}\,, \qquad \frac{\sigma_{p'}^{(-)}}{\sigma_{q'}^{(-)}} = \frac{\hat{Q}_1^*\hat{Q}_2}{\alpha_2\beta_1}\,. \tag{3.6.28}$$

Being solutions to a quadratic equation with real coefficients, they are complex conjugate of each other. Using now the second equation in (3.6.21), together with (3.6.25) and the on-shell conditions (3.6.9), we arrive at the following expression of the solution $\sigma_i^{(+)}$ to the SE

$$\begin{aligned}
\sigma_{p'}^{(+)} &= C_q\,\frac{(1+\beta_2)}{\beta_1}\hat{Q}_1, \\
\sigma_{q'}^{(+)} &= C_q\,\hat{Q}_2, \\
\sigma_{k}^{(+)} &= C_q\,\frac{(1+\beta_2)}{\beta_1-\beta_2}\left(\hat{Q}_1-\hat{Q}_2\right).
\end{aligned} \tag{3.6.29}$$

To fix the undetermined constant C_q we identify $\sigma_i^{(+)}$ with Fairlie's solution (3.6.11). This fixes C_q to be

$$C_q = \frac{e^{-i\theta_2}}{1+\beta_2}. \tag{3.6.30}$$

In order to understand the presence of the phase in this expression, we should point out that, in setting $\sigma_p = \infty$ and $\sigma_q = 0$, we only partially fixed the $SL(2,\mathbb{C})$ invariance of the moduli space of punctured spheres. This leaves us with complex rescalings as the residual invariance. We can make use of this freedom to set the phase of the constant C_q as in (3.6.30), which geometrically corresponds to a change in the origin of the azimuthal angles in the Riemann sphere. Our choice, which sets $\sigma_{q'}^{(\pm)}$ on the real axis, leads to a more symmetric form of the two solutions to the SE for the five-point amplitude

$$
\begin{aligned}
\sigma_{p'}^{(+)} &= \sigma_{p'}^{(-)*} = \frac{\hat{Q}_1 e^{-i\theta_2}}{\beta_1} = \sqrt{\frac{\alpha_1 - 1}{\beta_1}}\, e^{i(\theta_1 - \theta_2 + \pi)}, \\
\sigma_{q'}^{(+)} &= \sigma_{q'}^{(-)*} = \frac{\hat{Q}_2 e^{-i\theta_2}}{1+\beta_2} = \sqrt{\frac{\alpha_2}{1+\beta_2}}, \\
\sigma_{k}^{(+)} &= \sigma_{k}^{(-)*} = \frac{(\hat{Q}_1 - \hat{Q}_2) e^{-i\theta_2}}{\beta_1 - \beta_2} = \frac{\sqrt{(\alpha_1 - 1)\beta_1}\, e^{i(\theta_1 - \theta_2)} - \sqrt{\alpha_2(1+\beta_2)}}{\beta_1 - \beta_2}.
\end{aligned} \tag{3.6.31}
$$

Compared to the original result in Eq. (3.6.17), the improvement is obvious.

The localization of the punctures on the Riemann sphere can be also given in terms of the unit vectors

$$
\mathbf{u}_{p'}^{(\pm)} = \frac{1}{1-\alpha_1-\beta_1}\Big(-2\sqrt{(\alpha_1 - 1)\beta_1}\cos\gamma, \mp 2\sqrt{(\alpha_1 - 1)\beta_1}\sin\gamma, 1 - \alpha_1 + \beta_1 \Big), \tag{3.6.32}
$$

$$
\mathbf{u}_{q'}^{(\pm)} = \frac{1}{\alpha_2+\beta_2+1}\Big(2\sqrt{\alpha_2(1+\beta_2)}, 0, \alpha_2 - \beta_2 - 1 \Big), \tag{3.6.33}
$$

$$
\begin{aligned}
\mathbf{u}_{k}^{(\pm)} = \frac{1}{\alpha_1+\beta_1-\alpha_2-\beta_2}\Big(& 2\sqrt{(\alpha_1 - 1)\beta_1}\cos\gamma - \sqrt{\alpha_2(1+\beta_2)}, \\
& \mp 2\sqrt{(\alpha_1 - 1)\beta_1}\sin\gamma, \alpha_1 - \beta_1 - \alpha_2 + \beta_2 \Big),
\end{aligned} \tag{3.6.34}
$$

where we have defined $\gamma = \theta_1 - \theta_2$. As announced, $\sigma_i^{(+)}$ corresponds to Fairlie's solution, after choosing the origin of azimuthal angles such that $\theta_2 = 0$ in Eq. (3.6.12). The second solution $\sigma_i^{(-)}$ is obtained by reflecting the first one with respect to the $y = 0$ plane.

3.6.3 Kawai-Lewellen-Tye (KLT) orthogonality

Alternatively, it is also possible to obtain the solutions to the SE by exploiting the property of KLT orthogonality [62]. The idea here, in the same line as the previous derivation, would be to compute any desired solution using Fairlie's punctures as the starting point.

Recall that the KLT momentum kernel is a bilinear form from which one can construct the full n-graviton amplitude by just using two copies of gauge partial subamplitudes $A_{\mathrm{YM}}(1,\ldots,n)$. In more precise terms, we have

$$
M_n = \sum_{\alpha,\beta\in S_{n-3}} A_{\mathrm{YM}}(1,\alpha,n-1,n)\, S[\alpha|\beta]\, A_{\mathrm{YM}}(1,\beta,n,n-1)\,. \tag{3.6.35}
$$

The kernel is a $(n-3)! \times (n-3)!$-matrix, whose entries are $(n-3)^{\text{th}}$-degree homogeneous polynomials on the Mandelstam invariants and read[4]

$$S[\alpha|\beta] = \prod_{i=2}^{n-2} \left(s_{1,\alpha(i)} + \sum_{j=2}^{i-1} \theta(\alpha(j),\alpha(i))_\beta \, s_{\alpha(j),\alpha(i)} \right) , \tag{3.6.36}$$

with $\alpha, \beta \in S_{n-3}$, $\theta(i,j)_\beta = 1$ if the ordering of i,j is the same in both sequences of labels, $\alpha(2,\ldots,n-2)$ and $\beta(2,\ldots,n-2)$, and zero otherwise.

The basic idea behind KLT orthogonality would be the following. Given any two solutions, $\sigma^{(i)} = \{\sigma_1^{(i)},\ldots,\sigma_n^{(i)}\}$ and $\sigma^{(j)} = \{\sigma_1^{(j)},\ldots,\sigma_n^{(j)}\}$, one can construct an inner product of the form

$$(i,j) := \sum_{\alpha,\beta \in S_{n-3}} V^{(i)}(\alpha)\, S[\alpha|\beta]\, U^{(j)}(\beta) , \tag{3.6.37}$$

where $S[\alpha|\beta]$ is the momentum kernel described above and, $V^{(i)}$ and $U^{(j)}$ are $(n-3)!$-dimensional vectors defined[5] as

$$\begin{aligned} V^{(i)}(\omega) &= \frac{1}{(\sigma_1^{(i)} - \sigma_{\omega(2)}^{(i)})(\sigma_{\omega(2)}^{(i)} - \sigma_{\omega(3)}^{(i)}) \ldots (\sigma_{\omega(n-2)}^{(i)} - \sigma_{n-1}^{(i)})(\sigma_{n-1}^{(i)} - \sigma_n^{(i)})(\sigma_n^{(i)} - \sigma_1^{(i)})} , \\ U^{(j)}(\omega) &= \frac{1}{(\sigma_1^{(j)} - \sigma_{\omega(2)}^{(j)})(\sigma_{\omega(2)}^{(j)} - \sigma_{\omega(3)}^{(j)}) \ldots (\sigma_{\omega(n-2)}^{(j)} - \sigma_n^{(j)})(\sigma_n^{(j)} - \sigma_{n-1}^{(j)})(\sigma_{n-1}^{(j)} - \sigma_1^{(j)})} . \end{aligned} \tag{3.6.38}$$

It turns out that any two distinct solutions are said to be orthogonal with respect to this product if they satisfy the condition

$$\frac{(i,j)}{(i,i)^{\frac{1}{2}}(j,j)^{\frac{1}{2}}} = \delta_{ij} . \tag{3.6.39}$$

Therefore, taking Fairlie's solution, we are given a new set of constraints that may simplify the problem of finding the solutions of SE a little bit, e.g.

$$(F,i) = 0 , \qquad (i,F) = 0 . \tag{3.6.40}$$

The non-commutativity of the inner product comes from the asymmetry between $V(\omega)$ and $U(\omega)$.

Particularizing to the 5-point case, the KLT momentum kernel is a 2×2-matrix with

[4]Note that it corresponds to the field theory limit of the kernel used for closed strings. See Eq. (2.6.2) for details.

[5]Notice that both definitions differ on the relative order of the punctures σ_{n-1} and σ_n.

a quite simple structure in terms of Sudakov variables

$$S = \begin{pmatrix} s_{pq}(s_{pp'} + s_{p'q}) & s_{pq}s_{pp'} \\ s_{pq}s_{pp'} & s_{pp'}(s_{pq} + s_{qp'}) \end{pmatrix} = \begin{pmatrix} -1 + \alpha_1 + \beta_1 & \beta_1 \\ \beta_1 & \alpha_1\beta_1 \end{pmatrix} . \tag{3.6.41}$$

The vectors $V^{(F)}$ and $U^{(F)}$ are also easily computable by plugging Fairlie's solution (3.6.11) into their definition (3.6.38)

$$\begin{aligned} V^{(F)} &= \begin{pmatrix} \frac{\beta_1^2(\beta_1-\beta_2)(1+\beta_2)^2}{\hat{Q}_1\left(\hat{Q}_1(1+\beta_2)-\hat{Q}_2\beta_1\right)\left(\hat{Q}_1(1+\beta_2)-\hat{Q}_2(1+\beta_1)\right)} & \frac{\beta_1(\beta_1-\beta_2)(1+\beta_2)^2}{\hat{Q}_1\hat{Q}_2\left(\hat{Q}_1(1+\beta_2)-\hat{Q}_2(1+\beta_1)\right)} \end{pmatrix} \times \frac{1}{\sigma_p^2} , \\ U^{(F)} &= \begin{pmatrix} \frac{\beta_1^2(\beta_1-\beta_2)^2(1+\beta_2)}{\hat{Q}_1\left(\hat{Q}_1\beta_2-\hat{Q}_2\beta_1\right)\left(\hat{Q}_1(1+\beta_2)-\hat{Q}_2(1+\beta_1)\right)} & -\frac{\beta_1(\beta_1-\beta_2)^2(1+\beta_2)}{\hat{Q}_1\left(\hat{Q}_1-\hat{Q}_2\right)\left(\hat{Q}_1(1+\beta_2)-\hat{Q}_2(1+\beta_1)\right)} \end{pmatrix} \times \frac{1}{\sigma_p^2} . \end{aligned} \tag{3.6.42}$$

Notice that we have factorized out one of the punctures $\sigma_p^{(F)} \to \infty$ because it does not play any role in our constraints (3.6.40). Therefore, in order to find the second solution to the 5-point SE, we end up with the following system of equations

$$(F, i) = (i, F) = 0 \qquad \Rightarrow$$

$$\begin{cases} \hat{Q}_1(1+\beta_2)\left[\alpha_1(\sigma_{p'}^{(i)} - \sigma_k^{(i)}) + \sigma_k^{(i)}\right] + \hat{Q}_2(1-\alpha_1)\left[\sigma_k^{(i)} + \beta_1(\sigma_{p'}^{(i)} - \sigma_k^{(i)})\right] = 0 , \\ \hat{Q}_2(\alpha_1 - 1)\left[\beta_1\sigma_{p'}^{(i)} - (1+\beta_1)\sigma_{q'}\right] + \hat{Q}_1\left[(\beta_1 - \alpha_1\beta_2)\sigma_{p'}^{(i)} + (\alpha_1 - 1)(1+\beta_2)\sigma_{q'}^{(i)}\right] = 0 , \end{cases} \tag{3.6.43}$$

whose solution is immediate and given by

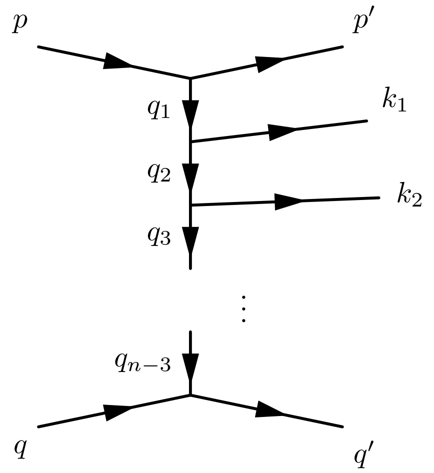
$$\begin{aligned} \frac{\sigma_{p'}^{(i)}}{\sigma_{q'}^{(i)}} &= \frac{(\alpha_1 - 1)\left[\hat{Q}_2(1+\beta_1) - \hat{Q}_1(1+\beta_2)\right]}{\hat{Q}_2(\alpha_1 - 1)\beta_1 + \hat{Q}_1(\beta_1 - \alpha_1\beta_2)} = \frac{\hat{Q}_1^*\hat{Q}_2}{\alpha_2\beta_1} = \left(\frac{\sigma_{p'}^{(F)}}{\sigma_{q'}^{(F)}}\right)^* , \\ \frac{\sigma_k^{(i)}}{\sigma_{q'}^{(i)}} &= \frac{\hat{Q}_2(\alpha_1 - 1)\beta_1 - \hat{Q}_1\alpha_1(1+\beta_2)}{\hat{Q}_1(\alpha_1 - 1)\beta_1 + \hat{Q}_1(\beta_1 - \alpha_1\beta_2)} = \frac{(\hat{Q}_1^* - \hat{Q}_2^*)\hat{Q}_2}{(\beta_1 - \beta_2)\alpha_2} = \left(\frac{\sigma_k^{(F)}}{\sigma_{q'}^{(F)}}\right)^* . \end{aligned} \tag{3.6.44}$$

As expected, it turns out to be the complex conjugate of Fairlie's solution.

3.7 The six-point case

Along the lines of what we did in the last two sections, we are considering now the scattering of 6 particles

$$p + q \to p' + k_1 + k_2 + q' . \tag{3.7.1}$$



$$p = \frac{\sqrt{s}}{2}(1,0,0,1)$$
$$q = \frac{\sqrt{s}}{2}(1,0,0,-1)$$
$$p' = p - q_1$$
$$q' = q + q_{n-3}$$
$$k_i = q_i - q_{i+1}$$

Figure 3.5: Sudakov parametrization of a general n-point process.

In this case, the mapping between particle momenta and punctures in the sphere given in Eq. (3.5.2) gets enlarged by

$$k_1^\mu = -\oint_{|z-\sigma_{k_1}|=\epsilon} \frac{\mathrm{d}z}{2\pi i}\,\omega^\mu(z)\;, \qquad k_2^\mu = -\oint_{|z-\sigma_{k_2}|=\epsilon} \frac{\mathrm{d}z}{2\pi i}\,\omega^\mu(z)\;, \tag{3.7.2}$$

and the meromorphic function $\omega^\mu(z)$ reads

$$\omega^\mu(z) = \frac{p^\mu}{z-\sigma_p} + \frac{q^\mu}{z-\sigma_q} - \frac{p'^\mu}{z-\sigma_{p'}} - \frac{k_1^\mu}{z-\sigma_{k_1}} - \frac{k_2^\mu}{z-\sigma_{k_2}} - \frac{q'^\mu}{z-\sigma_{q'}}\;. \tag{3.7.3}$$

Sudakov parameters can be introduced in the following way

$$p' = p - q_1\;, \qquad k_1 = q_1 - q_2\;, \qquad k_2 = q_2 - q_3\;, \qquad q' = q + q_3\;, \tag{3.7.4}$$

where $q_i \equiv \alpha_i p + \beta_i q + \mathbf{q_i}$ and again we are defining transverse momentum as

$$\mathbf{q_i} \equiv q_i^\perp(0, \cos\theta_i, \sin\theta_i, 0) \qquad \leftrightarrow \qquad Q_i \equiv q_i^\perp e^{i\theta_i}\;. \tag{3.7.5}$$

In the general n-point case, all momentum vectors would be parametrized as shown in Fig. 3.5. All the results from next section regarding the energy dependencies, onshell conditions or locations of the punctures are summarized in Appendix B.4 for arbitrary n.

3.7.1 Location of the punctures

Without writing again all explicit dependencies, it is easy to see that the energies of the outgoing particles in terms of Sudakov variables are

$$\begin{aligned} \omega_{p'} &= \frac{\sqrt{s}}{2}\,(1-\alpha_1-\beta_1)\ , & \omega_{k_1} &= \frac{\sqrt{s}}{2}\,(\alpha_1+\beta_1-\alpha_2-\beta_2)\ , \\ \omega_{q'} &= \frac{\sqrt{s}}{2}\,(1+\alpha_3+\beta_3)\ , & \omega_{k_2} &= \frac{\sqrt{s}}{2}\,(\alpha_2+\beta_2-\alpha_3-\beta_3)\ , \end{aligned} \tag{3.7.6}$$

and that direct computation over momentum vectors leads to these constraints coming from onshellness

$$\begin{aligned} (p')^2 = 0 \quad &\Rightarrow \quad |\hat{Q}_1|^2 = (\alpha_1-1)\beta_1\ , \\ k_1^2 = 0 \quad &\Rightarrow \quad |\hat{Q}_1-\hat{Q}_2|^2 = (\alpha_1-\alpha_2)(\beta_1-\beta_2)\ , \\ k_2^2 = 0 \quad &\Rightarrow \quad |\hat{Q}_2-\hat{Q}_3|^2 = (\alpha_2-\alpha_3)(\beta_2-\beta_3)\ , \\ (q')^2 = 0 \quad &\Rightarrow \quad |\hat{Q}_3|^2 = \alpha_3(1+\beta_3)\ . \end{aligned} \tag{3.7.7}$$

Comparing with the previous cases, we see that new variables are entering into the game —i.e. $\{\alpha_3, \beta_3, \hat{Q}_3\}$— but again not all of them are independent. Counting degrees of freedom[6], it turns out that there are 8 independent parameters for $n = 6$. In order to keep some symmetry, we can choose for example α_1, α_2, α_3, β_1, β_2, β_3, $q_2^\perp$, and the center-of-mass energy s, as our description variables. This being said, from the conditions in Eq. (3.7.7), we can write further identities which will be useful for later simplifications, e.g.

$$\begin{aligned} \hat{Q}_1\hat{Q}_2^* + \hat{Q}_1^*\hat{Q}_2 &= (\hat{q}_2^\perp)^2 + \beta_1(\alpha_1-1) - (\alpha_1-\alpha_2)(\beta_1-\beta_2)\ , \\ \hat{Q}_2\hat{Q}_3^* + \hat{Q}_2^*\hat{Q}_3 &= (\hat{q}_2^\perp)^2 + \alpha_3(1+\beta_3) - (\alpha_2-\alpha_3)(\beta_2-\beta_3)\ . \end{aligned} \tag{3.7.8}$$

A similar expression might be given for the remaining combination $\hat{Q}_1\hat{Q}_3^*+\hat{Q}_1^*\hat{Q}_3$; however, some degeneracies appear due to the quadratic nature of the on-shell conditions. Let us explore this fact in detail.

Since $\hat{Q}_i\hat{Q}_j^*+\hat{Q}_i^*\hat{Q}_j = 2|\hat{Q}_i||\hat{Q}_j|\cos(\theta_i-\theta_j)$, we could start by inspecting the following trigonometric identity

$$\cos\theta_{13} = \cos\theta_{12}\cos\theta_{23} + \sin\theta_{12}\sin\theta_{23}\ . \tag{3.7.9}$$

Although the *cosines* on the right-hand-side are perfectly determined by the on-shell conditions above, *sines* are given up to a sign. In order to get rid of this ambiguity, we

[6] For the general n-point case, we are left with $3n-10$ degrees of freedom after constraining the process by momentum conservation, onshellness and overall spatial azimuthal orientation.

are forced to use the formula

$$\sin^2\theta_{12}\sin^2\theta_{23} = (1-\cos^2\theta_{12})(1-\cos^2\theta_{23}) = (\cos\theta_{12}\cos\theta_{23}-\cos\theta_{13})^2\ . \qquad (3.7.10)$$

Coming back to Sudakov variables, after a little algebra, we have the following equation

$$\begin{aligned}
&\left(1-\frac{(|\hat{Q}_1|^2+|\hat{Q}_2|^2-\hat{Q}_{12}^2)^2}{4|\hat{Q}_1|^2|\hat{Q}_2|^2}\right)\left(1-\frac{(|\hat{Q}_2|^2+|\hat{Q}_3|^2-\hat{Q}_{23}^2)^2}{4|\hat{Q}_2|^2|\hat{Q}_3|^2}\right) = \\
&= \left[\frac{1}{2}\left(\frac{\hat{Q}_{13}^2}{|\hat{Q}_1||\hat{Q}_3|}-\frac{|\hat{Q}_1|}{|\hat{Q}_3|}-\frac{|\hat{Q}_3|}{|\hat{Q}_1|}\right)+\frac{1}{4}\left(\frac{\hat{Q}_{12}^2}{|\hat{Q}_1||\hat{Q}_2|}-\frac{|\hat{Q}_1|}{|\hat{Q}_2|}-\frac{|\hat{Q}_2|}{|\hat{Q}_1|}\right)\times\right. \\
&\qquad\left.\times\left(\frac{\hat{Q}_{23}^2}{|\hat{Q}_2||\hat{Q}_3|}-\frac{|\hat{Q}_2|}{|\hat{Q}_3|}-\frac{|\hat{Q}_3|}{|\hat{Q}_2|}\right)\right]^2 , \qquad (3.7.11)
\end{aligned}$$

which can be recast into a polynomial form as

$$(\hat{q}_2^\perp)^2\hat{Q}_{13}^4 + \left[(\hat{q}_2^\perp)^4+(\hat{q}_2^\perp)^2c_1+c_2\right]\hat{Q}_{13}^2 + \left[(\hat{q}_2^\perp)^2c_3+c_4\right] = 0\ , \qquad (3.7.12)$$

where the explicit form of the coefficients c_1, c_2, c_3 and c_4 in terms of the Sudakov variables are given in Appendix B.5. Also notice that we are using the notation $\hat{Q}_{ij} \equiv |\hat{Q}_i - \hat{Q}_j|$ for simplicity.

The important fact here is that the analytic expression of $\hat{Q}_{13}^2$ alone —and, consequently, that of $\hat{Q}_1\hat{Q}_3^* + \hat{Q}_1^*\hat{Q}_3$— is going to be a non-rational function of our Sudakov parameters

$$\hat{Q}_{13}^2 = -\frac{1}{2}\left[(\hat{q}_2^\perp)^2+c_1+\frac{c_2}{(\hat{q}_2^\perp)^2}\right] \pm \sqrt{\frac{1}{4}\left[(\hat{q}_2^\perp)^2+c_1+\frac{c_2}{(\hat{q}_2^\perp)^2}\right]^2-\left[c_3+\frac{c_4}{(\hat{q}_2^\perp)^2}\right]}\ . \qquad (3.7.13)$$

One might hope the term below the square root magically simplifies such that it can be written as a perfect square, but this is not the case. Actually it can be seen numerically by setting integer values into the Sudakov parameters and checking whether the square root remains there or not at the end. Therefore from now on, we will keep this dependency explicit in order to make this non-rationality manifest.

The coordinates of the punctures corresponding to Fairlie's solution and its complex conjugate are given by

$$\begin{aligned}
&\sigma_{p'}^{(1)} = \frac{\hat{Q}_1}{\beta_1}\ , \qquad \sigma_{k_1}^{(1)} = \frac{\hat{Q}_1-\hat{Q}_2}{\beta_1-\beta_2}\ , \qquad \sigma_{k_2}^{(1)} = \frac{\hat{Q}_2-\hat{Q}_3}{\beta_2-\beta_3}\ , \qquad \sigma_{q'}^{(1)} = \frac{\hat{Q}_3}{1+\beta_3}\ , \\
&\sigma_{p'}^{(2)} = \frac{\hat{Q}_1^*}{\beta_1}\ , \qquad \sigma_{k_1}^{(2)} = \frac{\hat{Q}_1^*-\hat{Q}_2^*}{\beta_1-\beta_2}\ , \qquad \sigma_{k_2}^{(2)} = \frac{\hat{Q}_2^*-\hat{Q}_3^*}{\beta_2-\beta_3}\ , \qquad \sigma_{q'}^{(2)} = \frac{\hat{Q}_3^*}{1+\beta_3}\ . \qquad (3.7.14)
\end{aligned}$$

Still, there is a total of six different solutions, whose finding will be the subject of the next section. Notice again that Fairlie's punctures can be understood as the stereographic projection of the unit vectors defining the direction of flight of the outgoing particles

$$\begin{aligned}
\mathbf{u}_{p'} &= \left(\frac{-2\, q_1^\perp \cos\theta_1}{\sqrt{s}\,(1-\alpha_1-\beta_1)}, \frac{-2\, q_1^\perp \sin\theta_1}{\sqrt{s}\,(1-\alpha_1-\beta_1)}, \frac{1-\alpha_1+\beta_1}{1-\alpha_1-\beta_1} \right) ,\\
\mathbf{u}_{k_1} &= \left(\frac{2\,(q_1^\perp \cos\theta_1 - q_2^\perp \cos\theta_2)}{\sqrt{s}\,(\alpha_1+\beta_1-\alpha_2-\beta_2)}, \frac{2\,(q_1^\perp \sin\theta_1 - q_2^\perp \sin\theta_2)}{\sqrt{s}\,(\alpha_1+\beta_1-\alpha_2-\beta_2)}, \frac{\alpha_1-\beta_1-\alpha_2+\beta_2}{\alpha_1+\beta_1-\alpha_2-\beta_2} \right) ,\\
\mathbf{u}_{k_2} &= \left(\frac{2\,(q_2^\perp \cos\theta_2 - q_3^\perp \cos\theta_3)}{\sqrt{s}\,(\alpha_2+\beta_2-\alpha_3-\beta_3)}, \frac{2\,(q_2^\perp \sin\theta_2 - q_3^\perp \sin\theta_3)}{\sqrt{s}\,(\alpha_2+\beta_2-\alpha_3-\beta_3)}, \frac{\alpha_2-\beta_2-\alpha_3+\beta_3}{\alpha_2+\beta_2-\alpha_3-\beta_3} \right) ,\\
\mathbf{u}_{q'} &= \left(\frac{2\, q_3^\perp \cos\theta_3}{\sqrt{s}\,(\alpha_3+\beta_3+1)}, \frac{2\, q_3^\perp \sin\theta_3}{\sqrt{s}\,(\alpha_3+\beta_3+1)}, \frac{\alpha_3-\beta_3-1}{\alpha_3+\beta_3+1} \right) ,
\end{aligned} \tag{3.7.15}$$

whose projection onto the equatorial plane lie on circumferences of radii

$$\begin{aligned}
R_{p'} &= 2\sqrt{\frac{|\hat{Q}_1|^2}{(1-\alpha_1-\beta_1)^2}} = 2\sqrt{\frac{(\alpha_1-1)\beta_1}{(1-\alpha_1-\beta_1)^2}} ,\\
R_{k_1} &= 2\sqrt{\frac{|\hat{Q}_1-\hat{Q}_2|^2}{(\alpha_1+\beta_1-\alpha_2-\beta_2)^2}} = 2\sqrt{\frac{(\alpha_1-\alpha_2)(\beta_1-\beta_2)}{(\alpha_1+\beta_1-\alpha_2-\beta_2)^2}} ,\\
R_{k_2} &= 2\sqrt{\frac{|\hat{Q}_2-\hat{Q}_3|^2}{(\alpha_2+\beta_2-\alpha_3-\beta_3)^2}} = 2\sqrt{\frac{(\alpha_2-\alpha_3)(\beta_2-\beta_3)}{(\alpha_2+\beta_2-\alpha_3-\beta_3)^2}} ,\\
R_{q'} &= 2\sqrt{\frac{|\hat{Q}_3|^2}{(1+\alpha_3+\beta_3)^2}} = 2\sqrt{\frac{\alpha_3(1+\beta_3)}{(1+\alpha_3+\beta_3)^2}} .
\end{aligned} \tag{3.7.16}$$

3.7.2 Scattering Equations

The $n = 6$ SE in their standard form [see Eq. (3.2.7)] make use of all the two-point kinematic invariants. Written in terms of Sudakov parameters, some of them get fully characterized and read as

$$\begin{aligned}
s_{pq} &= s , & s_{pp'} &= s\beta_1 ,\\
s_{pk_1} &= s(\beta_2-\beta_1) , & s_{pk_2} &= s(\beta_3-\beta_2) ,\\
s_{pq'} &= -s(1+\beta_3) , & s_{qp'} &= s(\alpha_1-1) ,\\
s_{qk_1} &= s(\alpha_2-\alpha_1) , & s_{qk_2} &= s(\alpha_3-\alpha_2) ,\\
s_{qq'} &= -s\alpha_3 , & s_{p'k_1} &= s(\alpha_2-1)\beta_2-(q_2^\perp)^2 ,\\
s_{k_2q'} &= s\alpha_2(1+\beta_2)-(q_2^\perp)^2 .
\end{aligned} \tag{3.7.17}$$

Nevertheless, there are few of them depending explicitly on $|Q_1 - Q_3|^2$, thus carrying non-rational terms, whose particular expressions are

$$\begin{aligned}
s_{p'k_2} &= -s(\alpha_3+\beta_3)+s\beta_2(1-\alpha_2)+(q_2^\perp)^2-s(\alpha_1-\alpha_3)(\beta_1-\beta_3)+Q_{13}^2\ ,\\
s_{p'q'} &= s(-1+\alpha_1-\alpha_3)(-1+\beta_1-\beta_3)-Q_{13}^2\ ,\\
s_{k_1k_2} &= s(\alpha_1-\alpha_3)(\beta_1-\beta_3)-Q_{13}^2\ ,\\
s_{k_1q'} &= s(\alpha_1+\beta_1-\alpha_2-\alpha_2\beta_2)+(q_2^\perp)^2-s(\alpha_1-\alpha_3)(\beta_1-\beta_3)+Q_{13}^2\ .
\end{aligned} \tag{3.7.18}$$

We know that for the six-point case there are three independent SE with six inequivalent solutions. The complete set of equations in terms of Sudakov variables are given in Appendix B.6. Of course, Fairlie's punctures and their complex conjugates leave us with the remaining four solutions to be determined. After a few attempts, it makes evident that getting rid of all the denominators makes things easier. Formally, the polynomial representation of the SE was first presented in [84], and can be expressed in a compact notation as[7]

$$\mathcal{H}_m(\sigma) \equiv \sum_{\substack{|S|=m \\ S\subset\{1,\ldots,n\}}} p_S^2\sigma_S = 0 \qquad \text{for} \qquad 2\le m\le n-2\ , \tag{3.7.19}$$

where $p_S = \sum_{i\in S} p_i$ and $\sigma_S = \prod_{i\in S}\sigma_i$. Notice that the representation is simply a rearrangement of the standard one and is valid thanks to the fact that $\sigma_i = \sigma_j$ is not allowed inside the physical region for any pair of particles i and j.

We are going to need as well the three-point kinematic invariants

$$\begin{aligned}
& s_{pqp'} = s_{q'k_1k_2} = \alpha_1+\beta_1\ ,\\
& s_{pqq'} = s_{p'k_1k_2} = -(\alpha_3+\beta_3)\ ,\\
& s_{pqk_1} = s_{p'q'k_2} = 1-\alpha_1+\alpha_2-\beta_1+\beta_2\ ,\\
& s_{pqk_2} = s_{p'q'k_1} = 1-\alpha_2+\alpha_3-\beta_2+\beta_3\ ,\\
& s_{pp'q'} = s_{qk_1k_2} = -\hat{Q}_{13}^2+(\alpha_1-\alpha_3)(-1+\beta_1-\beta_3)\ ,\\
& s_{pp'k_1} = s_{qq'k_2} = -(\hat{q}_2^\perp)^2+\alpha_2\beta_2\ ,\\
& s_{pp'k_2} = s_{qq'k_1} = \hat{Q}_{13}^2+(\hat{q}_2^\perp)^2+\beta_1-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3(-1+\beta_1-\beta_3)+\alpha_1\beta_3\ ,\\
& s_{pq'k_1} = s_{qp'k_2} = -1+\hat{Q}_{13}^2+(\hat{q}_2^\perp)^2+\alpha_3\beta_1+\beta_2-\alpha_2(1+\beta_2)-(1+\alpha_3)\beta_3\\
& \qquad\qquad\qquad +\alpha_1(1-\beta_1+\beta_3)\ ,\\
& s_{pq'k_2} = s_{qp'k_1} = -(\hat{q}_2^\perp)^2+(-1+\alpha_2)(1+\beta_2)\ ,\\
& s_{pk_1k_2} = s_{qp'q'} = -\hat{Q}_{13}^2+(-1+\alpha_1-\alpha_3)(\beta_1-\beta_3)\ .
\end{aligned} \tag{3.7.20}$$

[7] $\mathcal{H}_1$, $\mathcal{H}_{n-1}$ and $\mathcal{H}_n$ identically vanish due to momentum conservation and the on-shell conditions.

Thus, after partial fixing the $SL(2,\mathbb{C})$ invariance $\sigma_p \to \infty$ and $\sigma_q \to 0$ in Eq. (3.7.19), the SE turn out to be equivalent to

$$\begin{aligned}
\sigma_{p'}\beta_1 - \sigma_{q'}(1+\beta_3) - \sigma_{k_1}(\beta_1-\beta_2) - \sigma_{k_2}(\beta_2-\beta_3) &= 0\ , \\
\sigma_{p'}\sigma_{q'}\left[\hat{Q}_{13}^2 - (\alpha_1-\alpha_3)(-1+\beta_1-\beta_3)\right] & \\
+\sigma_{p'}\sigma_{k_1}\left[(\hat{q}_2^{\perp})^2 - \alpha_2\beta_2\right] & \\
-\sigma_{p'}\sigma_{k_2}\left[\hat{Q}_{13}^2 + (\hat{q}_2^{\perp})^2 + \beta_1 - \alpha_1\beta_1 - \alpha_2\beta_2 + \alpha_3(-1+\beta_1-\beta_3) + \alpha_1\beta_3\right] & \\
-\sigma_{q'}\sigma_{k_1}\left[-1 + \hat{Q}_{13}^2 + (\hat{q}_2^{\perp})^2 + \alpha_3\beta_1 + \beta_2 - \alpha_2(1+\beta_2)\right. & \\
\left. -(1+\alpha_3)\beta_3 + \alpha_1(1-\beta_1+\beta_3)\right] &= 0\ , \\
\sigma_{p'}\sigma_{q'}\sigma_{k_1}(\alpha_2-\alpha_3) + \sigma_{p'}\sigma_{q'}\sigma_{k_2}(\alpha_1-\alpha_2) + \sigma_{p'}\sigma_{k_1}\sigma_{k_2}\alpha_3 + \sigma_{q'}\sigma_{k_1}\sigma_{k_2}(1-\alpha_1) &= 0\ .
\end{aligned} \tag{3.7.21}$$

Here we have a homogeneous system of equations on the σ_i variables, where the degree of each one progresively increases from the first to the last. It seems natural to start solving the first equation by substitution and proceed recursively throughout all of them to end up with a single equation on just one of the variables. The six-point case is pretty straightforward, whose systematic approach can be found in [85] for general n.

Therefore, picking for example $\sigma_{p'}$ as the initial variable, we end up with a 6^{th}-degree equation of the form

$$a_6\,\sigma_{p'}^6 + a_5\,\sigma_{p'}^5\sigma_{q'} + a_4\,\sigma_{p'}^4\sigma_{q'}^2 + a_3\,\sigma_{p'}^3\sigma_{q'}^3 + a_2\,\sigma_{p'}^2\sigma_{q'}^4 + a_1\,\sigma_{p'}\sigma_{q'}^5 + a_0\,\sigma_{q'}^6 = 0\ , \tag{3.7.22}$$

or equivalently (normalizing coefficients as $a_i' = a_i/a_6$)

$$\left(\frac{\sigma_{p'}}{\sigma_{q'}}\right)^6 + a_5'\left(\frac{\sigma_{p'}}{\sigma_{q'}}\right)^5 + a_4'\left(\frac{\sigma_{p'}}{\sigma_{q'}}\right)^4 + a_3'\left(\frac{\sigma_{p'}}{\sigma_{q'}}\right)^3 + a_2'\left(\frac{\sigma_{p'}}{\sigma_{q'}}\right)^2 + a_1'\left(\frac{\sigma_{p'}}{\sigma_{q'}}\right) + a_0' = 0. \tag{3.7.23}$$

where the coefficients a_i' just depend on the Sudakov parameters. Everything is consistent, since from this equation one can recover the $(n-3)! \overset{n=6}{\longrightarrow} 6$ different solutions to the original SE. The ratio $(\sigma_{p'}/\sigma_{q'})$ is just a reminder of the freedom we still have to fix the remaining $SL(2,\mathbb{C})$ symmetry of the problem.

We have not written explicitly the expressions for the coefficients because they turn out to be quite messy, spoiling the immediate attempt of solving the equation; however, information can be already extracted directly from some of them. In particular, a_0', not only is sufficiently simple to be written down, but gives the product of all solutions

$$a_0' \equiv \prod_{i=1}^{6} \frac{\sigma_{p'}^{(i)}}{\sigma_{q'}^{(i)}} = \frac{(\alpha_1-1)^2(1+\beta_3)^2}{\beta_1^2\alpha_3^2}\left[\frac{(q_2^\perp)^2-(-1+\alpha_2)(1+\beta_2)}{(q_2^\perp)^2-\alpha_2\beta_2}\right]$$

$$\times\left[\frac{-1+|\hat{Q}_1-\hat{Q}_3|^2+(q_2^\perp)^2+\alpha_3\beta_1+\beta_2-\alpha_2(1+\beta_2)-(1+\alpha_3)\beta_3+\alpha_1(1-\beta_1+\beta_3)}{|\hat{Q}_1-\hat{Q}_3|^2+(q_2^\perp)^2+\beta_1-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3(-1+\beta_1-\beta_3)+\alpha_1\beta_3}\right]$$

$$= \frac{s_{qp'}^2 s_{pq'}^2}{s_{pp'}^2 s_{qq'}^2}\,\frac{s_{pq'k_2}}{s_{qq'k_2}}\,\frac{s_{qp'k_2}}{s_{pp'k_2}}\,. \quad (3.7.24)$$

Notice that some of the first factors reproduce both Fairlie's solution and its complex conjugate as expected i.e.

$$\frac{\sigma_{p'}^{(F)}}{\sigma_{q'}^{(F)}}\frac{(\sigma_{p'}^{(F)})^*}{(\sigma_{q'}^{(F)})^*} = \frac{\hat{Q}_1\hat{Q}_1^*}{\beta_1^2}\frac{(1+\beta_3)^2}{\hat{Q}_3\hat{Q}_3^*} = \frac{(\alpha_1-1)(1+\beta_3)}{\beta_1\alpha_3}\,. \quad (3.7.25)$$

The other four solutions would give rise to the remaining factors in Eq. (3.7.24). Similarly, we can obtain the rest of the punctures by deriving the analogous of Eq. (3.7.23) for a different variable

$$\prod_{i=1}^{6}\frac{\sigma_{k_1}^{(i)}}{\sigma_{q'}^{(i)}} = \frac{(\alpha_1-\alpha_2)^2(1+\beta_3)^2}{(\beta_1-\beta_2)^2\alpha_3^2}\left[\frac{|\hat{Q}_1-\hat{Q}_3|^2-(\alpha_1-\alpha_3)(-1+\beta_1-\beta_3)}{|\hat{Q}_1-\hat{Q}_3|^2-(-1+\alpha_1-\alpha_3)(\beta_1-\beta_3)}\right]$$

$$\times\left[\frac{-(q_2^\perp)^2+(-1+\alpha_2)(1+\beta_2)}{-(q_2^\perp)^2+\alpha_2\beta_2}\right] = \frac{s_{qk_1}^2 s_{pq'}^2}{s_{pk_1}^2 s_{qq'}^2}\,\frac{s_{pp'q'}}{s_{qp'q'}}\,\frac{s_{qp'k_1}}{s_{pp'k_1}}\,, \quad (3.7.26)$$

$$\prod_{i=1}^{6}\frac{\sigma_{k_2}^{(i)}}{\sigma_{q'}^{(i)}} = \frac{(\alpha_2-\alpha_3)^2(1+\beta_3)^2}{(\beta_2-\beta_3)^2\alpha_3^2}\left[\frac{|\hat{Q}_1-\hat{Q}_3|^2-(\alpha_1-\alpha_3)(-1+\beta_1-\beta_3)}{|\hat{Q}_1-\hat{Q}_3|^2-(-1+\alpha_1-\alpha_3)(\beta_1-\beta_3)}\right]$$

$$\times\left[\frac{-1+|\hat{Q}_1-\hat{Q}_3|^2+(q_2^\perp)^2-\alpha_2+\alpha_3\beta_1+\beta_2-\alpha_2\beta_2-\alpha_1(-1+\beta_1-\beta_3)-\beta_3-\alpha_3\beta_3}{|\hat{Q}_1-\hat{Q}_3|^2+(q_2^\perp)^2+\beta_1-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3(-1+\beta_1-\beta_3)+\alpha_1\beta_3}\right]$$

$$= \frac{s_{qk_2}^2 s_{pq'}^2}{s_{pk_2}^2 s_{qq'}^2}\,\frac{s_{pp'q'}}{s_{qp'q'}}\,\frac{s_{qp'k_2}}{s_{pp'k_2}}\,. \quad (3.7.27)$$

It is important to point out that, although we did not manage to solve completely the equations, the use of Sudakov variables has been crucial to perform all the suitable simplifications needed to finally get the expressions in Eqs. (3.7.24), (3.7.26) and (3.7.27). Moreover, the presence of $|\hat{Q}_1-\hat{Q}_3|^2$ suggests that the remaining solutions are non-rational functions of the Sudakovs. This statement is in agreement with [66], where Fairlie's solution and its complex conjugate are described as the only rational ones.

3.8 Closing remarks

We have presented a first analysis of the use of Sudakov variables in the context of the SE formalism. Thanks to the decomposition between longitudinal and transverse components for particle momenta inherent to the Sudakov parametrization, we have been able to interpret Fairlie's punctures simply as their stereographic projection onto the unit sphere. Using these variables and a particular reference frame for the two incoming particles, it is possible to identify this solution to the SE as a choice of punctures on the Riemann sphere parametrized by the rapidity and the azimuthal angle —defined on the transverse plane to the collision axis of the incoming particles— of each particle. The punctures for the emitted particles are then living on circles parametrized by one Sudakov variable in the four-point case, and four Sudakov variables for five-particle amplitudes.

Whereas for the five-point SE we have seen how the use of Sudakov variables greatly facilitates any attempt to find the solutions in the naive way, the six point case —involving a higher number of solutions— entails a first challenge and a more subtle strategy is needed. We have shown first the general expression of Fairlie's punctures. After translating the problem into a polynomial form, in spite of not solving it completely, we managed to obtain at least enough information about the modulus of the punctures for all the remaining solutions, supporting the idea that the Sudakov parametrization is the natural language in this formalism. Different algebraic methods may still be exploited such as KLT orthogonality [62] or the helicity-sector decomposition of the SE [86], but we defer them to future work.

What we can do however, already with the results obtained in Eqs. (3.7.24), (3.7.26) and (3.7.27), is to study the behavior of the punctures in *multi-Regge-kinematics* (MRK) [87]. Having an n-point process of the form

$$p + q \quad \longrightarrow \quad p' + k_1 + k_2 + \ldots + q' \,, \tag{3.8.1}$$

it would be defined by the following condition over the rapidities of the emitted particles

$$Y_{p'} \gg Y_{k_1} \gg Y_{k_2} \gg \ldots \gg Y_{q'} \,. \tag{3.8.2}$$

Using a different language, in the MRK limit all transverse momenta are of the same order and much smaller than any other energy scale in the process, and longitudinal momenta are strongly ordered according to Eq. (3.8.2).

The study of this limit is specially simple in Sudakov variables, translating straightforwardly the condition in Eq. (3.8.2) into

$$\begin{aligned} 1 \gg \alpha_1 \gg \alpha_2 \gg \ldots \gg \alpha_{n-3} \sim \left(\hat{q}_i^{\perp}\right)^2 \,, \\ 1 \gg |\beta_{n-3}| \gg |\beta_{n-2}| \gg \ldots \gg |\beta_1| \sim \left(\hat{q}_i^{\perp}\right)^2 \,. \end{aligned} \tag{3.8.3}$$

In Refs. [68, 88] it is conjectured that both real and imaginary parts of the punctures are strongly ordered in the same way as the rapidities for any of the solutions. However, from the very definition of Fairlie's solution —i.e. $\sigma_i^{(F)} = e^{Y_i} e^{i\phi_i}$—, the statement can be challenged and reformulated to be only valid on the modulus of the punctures. Since these are already computed in Eqs. (3.7.24) (3.7.26) and (3.7.27), it is easy to check that

$$\begin{aligned}
\prod_{i=1}^{6} \frac{\sigma_{p'}^{(i)}}{\sigma_{q'}^{(i)}}\bigg|_{\mathrm{MRK}} &= \frac{s_{qp'}^2 s_{pq'}^2}{s_{pp'}^2 s_{qq'}^2} \frac{s_{pq'k_2}}{s_{qq'k_2}} \frac{s_{qp'k_2}}{s_{pp'k_2}}\bigg|_{\mathrm{MRK}} \approx \frac{1}{\beta_1^2 \alpha_3^2} \frac{1}{(\hat{q}_2^{\perp})^2} \frac{-1}{\alpha_1 \beta_3} , \\
\prod_{i=1}^{6} \frac{\sigma_{k_1}^{(i)}}{\sigma_{q'}^{(i)}}\bigg|_{\mathrm{MRK}} &= \frac{s_{qk_1}^2 s_{pq'}^2}{s_{pk_1}^2 s_{qq'}^2} \frac{s_{pp'q'}}{s_{qp'q'}} \frac{s_{qp'k_1}}{s_{pp'k_1}}\bigg|_{\mathrm{MRK}} \approx \frac{\alpha_1^2}{\beta_2^2 \alpha_3^2} \frac{-\alpha_1}{\beta_3} \frac{1}{(q_2^{\perp})^2} , \\
\prod_{i=1}^{6} \frac{\sigma_{k_2}^{(i)}}{\sigma_{q'}^{(i)}}\bigg|_{\mathrm{MRK}} &= \frac{s_{qk_2}^2 s_{pq'}^2}{s_{pk_2}^2 s_{qq'}^2} \frac{s_{pp'q'}}{s_{qp'q'}} \frac{s_{qp'k_2}}{s_{pp'k_2}}\bigg|_{\mathrm{MRK}} \approx \frac{\alpha_2^2}{\beta_3^2 \alpha_3^2} \frac{-\alpha_1}{\beta_3} \frac{-1}{\alpha_1 \beta_3} ,
\end{aligned} \tag{3.8.4}$$

which leads directly to the following behavior

$$\left(\prod_{i=1}^{6} \frac{\sigma_{p'}^{(i)}}{\sigma_{q'}^{(i)}} \right) \gg \left(\prod_{i=1}^{6} \frac{\sigma_{k_1}^{(i)}}{\sigma_{q'}^{(i)}} \right) \gg \left(\prod_{i=1}^{6} \frac{\sigma_{k_2}^{(i)}}{\sigma_{q'}^{(i)}} \right) . \tag{3.8.5}$$

Of course, this expression does not prove our reformulation of the conjecture, although it gives strong evidence of its validity, being in agreement with

$$\left|\sigma_{p'}^{(i)}\right| \gg \left|\sigma_{k_1}^{(i)}\right| \gg \left|\sigma_{k_2}^{(i)}\right| \gg \left|\sigma_{q'}^{(i)}\right| \qquad \forall i . \tag{3.8.6}$$

In view of this result, we may ask whether the SE factorize into radial and angular part in the MRK limit. A rigurous study of the MRK behavior of the SE is in progress.

In any case, apart from taking advantage of the Sudakov representation in the attempt at unravelling the analytical structure behind the SE solutions for general multiplicity n, this chapter can be considered as a background and first approach to the following one, where the CHY representation of scattering amplitudes will be proven to be really useful to complete the characterization of planar radiation zeros from Chapters 1 and 2.

Chapter 4

Sudakov Representation of CHY Amplitudes

4.1 Introduction

In the early 2000s, some important and original results were presented concerning a new representation of scattering amplitudes in contrast to the traditional Quantum Field Theory methods. First, Witten suggested in [89] a novel way of writing the tree-level S-matrix of Yang-Mills theory in four dimensions as an integral formula inspired in twistor space techniques and String Theory, which was later validated from momentum space in [90]. A few years later, it was found that the same representation was also applicable to gravity [91, 92] and even generalizable to arbitrary dimensions for both of them [63]. From there, further developments rapidly spread into a wider range of theories and scenarios [64, 93–95]. One may wonder then: what is the full set of theories that can be expressed in this integral representation? What are the main principles in which this representation is actually based on, that allow the description to be sort of universal and valid for all of them? The Cachazo-He-Yuan (CHY) formalism [62–64] has become the standard framework to tackle these questions.

More concretely, tree-level n-point scattering amplitudes are expressed as $(n-3)$-dimensional integrals over the moduli space of certain rational maps studied in detail in Chapter 3 under the name of *scattering equations* (SE) [63, 71, 72, 74, 83]. These equations are the key ingredient of the formalism and define a map from the space of kinematic invariants s_{ij} for n on-shell massless particles to the moduli space of n-punctured Riemann spheres. The general structure of the amplitude is

$$\mathcal{A}_n = \int \frac{\mathrm{d}^n z}{\mathrm{vol}[SL(2,\mathbb{C})]} \prod_{i=1}^{n}{}' \delta \left(\sum_{j=1\neq i}^{n} \frac{s_{ij}}{z_i - z_j} \right) \mathcal{I}(z,p,\varepsilon) \, , \tag{4.1.1}$$

where the integrand $\mathcal{I}(z,p,\varepsilon)$ gathers information about the interactions present in the particular process we are working with and therefore is theory-dependent, and the integration domain encodes the singularities arising from the kinematics of a general n-point collision and is theory-independent. A proof of this integral formula was given in [96] for arbitrary n in Yang-Mills theory.

There are some advantages supporting the success of this representation. Since the SE are objects already accounting for the whole singularity structure of the amplitude, it means that the integrands $\mathcal{I}(z, p, \varepsilon, \ldots)$ would be simply polynomial functions of the momentum vectors p^μ, polarization vectors ε^μ or other quantum numbers the theory may have. This was first conjectured in [97] and has been tested to be valid for all the theories described inside the formalism until now. Moreover, the evaluation of the integrand on the solutions of the SE through the Dirac delta functions $\prod_i \delta(\mathcal{S}_i)$, leads to a sum over gauge invariant quantities, in contrast to the usual Feynman diagram decomposition, getting rid of all the excess of information and reducing significantly the number of terms. In other words, we get a shorter and gauge invariant representation of scattering amplitudes at the cost of losing manifest locality and unitarity. Last but not least, it is always nice to have a uniform description in which some of the hidden symmetries that might exist on a theory become manifest, as it is the case of color-kinematics duality.

Nevertheless, the main difficulty with the CHY strategy is that the number of integrals defining the n-point S-matrix elements, although being compensated by the delta functions, still grows very rapidly. The reason is that the number of integrations to be carried out scales like the number of solutions to the SE, which is $(n-3)!$ for the n-point amplitude. There is also not a systematic way to analytically solve them for an arbitrary number of external particles apart from some specific regimes refered to in the previous chapter. As a consequence, the representation becomes increasingly inaccesible at high multiplicity. Despite that, there has been steady progress in the understanding of the solutions to the SE and the calculation of amplitudes in the formalism (see, for example, [65, 66, 84, 85, 98–101]).

The CHY proposal for the calculation of tree-level scattering amplitudes has an interpretation in terms of ambitwistor strings [76, 80, 81, 102–104] defined on a Riemann surface at genus zero. At loop level, supergravity integrands of four-point amplitudes at one and two loops have been obtained when introducing higher genus [76, 77, 79, 105]. Other connections to string theory amplitudes can be found, for example, in [106–108].

The outline of the chapter is as follows: in Section 4.2 we first review the general basics of the CHY framework. The integral representation of n-point tree-level scattering amplitudes for massless particles is studied in detail as well as some physical insights regarding the analytical structure of the integrands for a generic theory. Being the SE the backbone of the formalism, we sum up the Sudakov representation of the $n = 4, 5$ SE solutions from the previous chapter and connect them to the helicity sectors of the amplitude. Section 4.3 presents one of the simplest cases of a theory one can think of —i.e. scalar φ^3 theory— where Sudakov variables actually play an special role allowing for compact formulas of the four- and five-point amplitudes. Section 4.4 is devoted then to further investigate the main building blocks from which to construct more complicated integrands. In particular, we use color-kinematics duality to write down the amplitudes in the biadjoint scalar theory, Yang-Mills and gravity. Gluon and graviton emissions are studied also in Section 4.5 as a particular case of the former. Finally, as one of the main goals of this chapter, we see in Section 4.6 how the characterization of planar radiation zeros in Chapters 1 and 2 can be recovered from the CHY formalism when taking the appropriate limits in the Sudakov parametrization space, shedding at the same time some light over their nature and behavior.

4.2 CHY amplitudes formalism: review

4.2.1 Amplitude representation

The general structure of an n-point tree-level scattering amplitude can be expressed as an $(n-3)$-dimensional complex contour integral over the moduli space of n-punctured spheres on the support of the SE in this way

$$\mathcal{A}_n = \int \frac{\mathrm{d}^n z}{\mathrm{vol}[SL(2,\mathbb{C})]} \prod_{i=1}^{n}{}' \delta\left(\sum_{j=1\neq i}^{n} \frac{s_{ij}}{z_i - z_j}\right) \mathcal{I}(z,p,\varepsilon) \ . \tag{4.2.1}$$

Notice that the $SL(2,\mathbb{C})$ invariance present on the Riemann sphere forces to mod-out the symmetry group from the integration. On one side, we saw that there are only $n-3$ independent equations out of the whole set of SE. Therefore, in order to restrict the evaluation over inequivalent solutions, it is required to consider

$$\prod_{i}{}' \delta\left(\sum_{j\neq i}^{n} \frac{s_{ij}}{z_i - z_j}\right) := z_{ab}z_{bc}z_{ca} \prod_{i\neq a,b,c} \delta\left(\sum_{j=1\neq i}^{n} \frac{s_{ij}}{z_i - z_j}\right) \tag{4.2.2}$$

as the correct integration contour. Similarly on the other side, the prior fixing of three of the delta functions makes necessary to remove three integration variables

$$\frac{\mathrm{d}^n z}{\mathrm{vol}[SL(2,\mathbb{C})]} \equiv \frac{\prod_i \mathrm{d}z_i}{\mathrm{vol}[SL(2,\mathbb{C})]} := z_{ab}z_{bc}z_{ca} \prod_{i\neq a,b,c} \mathrm{d}z_i \ . \tag{4.2.3}$$

Both explicit definitions in Eqs. (4.2.2) and (4.2.3) are introduced in such a way that the whole measure is $SL(2,\mathbb{C})$ covariant and permutation invariant.

The definition of the amplitude in Eq. (4.2.1), despite being rather general, can be used to derive some of the constraints the integrand $\mathcal{I}(z,p,\varepsilon,\ldots)$ has to fulfill so that it is a well-defined and meaningful object for the representation. First of all, remember that the SE define a map that resolves the whole singular behavior of the amplitude. As a consequence, every single variable provides the integrand with a polynomial structure —excluding the z_i's, which give rise to all the poles after integration—. Moreover, taking into account the Lorentz invariance of the amplitude, it is clear that the full integral should be a $SL(2,\mathbb{C})$-invariant object in the new space of punctured spheres. Let us consider the general transformation described in Eq. (3.2.3)

$$z_i \mapsto z_i' = \frac{Az_i + B}{Cz_i + D} \quad \text{with} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{C}) \ . \tag{4.2.4}$$

It is straightforward to see that the integration measure[1] transforms as

$$\mathrm{d}^n z \quad \mapsto \quad \mathrm{d}^n z \prod_{i=1}^{n} (Cz_i + D)^{-2} . \tag{4.2.5}$$

Similarly, the delta functions transform in the following way

$$\prod_i{}' \, \delta \left(\sum_{i=1 \neq j}^{n} \frac{s_{ij}}{z_i - z_j} \right) \quad \mapsto \quad \prod_i{}' \, \delta \left(\sum_{i=1 \neq j}^{n} \frac{s_{ij}}{z_i - z_j} \right) \prod_{i=1}^{n} (Cz_i + D)^{-2} . \tag{4.2.6}$$

Therefore, the integrand must satisfy the scaling

$$\mathcal{I}(z, p, \varepsilon) \quad \mapsto \quad \mathcal{I}(z, p, \varepsilon) \prod_{i=1}^{n} (Cz_i + D)^{4} . \tag{4.2.7}$$

From here, it would be possible to start plugging different polynomial integrands in with the correct transformation law and to check whether they match with an existing theory or give rise to a new well-defined amplitude. In the next subsections, we will present some of the most common building blocks that are in fact used to construct many different theories. We will show accordingly how the Sudakov parametrization allows for notorious simplifications and rather compact analytic expressions for the four- and five-point amplitudes.

4.2.2 Maximally-Helicity-Violating & Fairlie's solution to the SE

Given the general integral structure of the amplitude, it can be handy for later purposes to summarize some of the main results concerning the SE and their solutions in the $n = 4$ and $n = 5$ cases.

The standard form of the SE, mapping the null light-cone in momentum space into the moduli space of n-punctured spheres, is

$$\mathcal{S}_i(\sigma) \equiv \sum_{j \neq i}^{n} \frac{s_{ij}}{\sigma_i - \sigma_j} = 0 \qquad \text{for} \qquad i = 1, \ldots, n . \tag{4.2.8}$$

Due to the $SL(2, \mathbb{C})$ redundancy, only $n - 3$ of them are independent and the space of solutions amounts to a total of $(n - 3)!$ inequivalent points. In general, obtaining the complete exact analytic full set of solutions is a hard task; however, it is known that Fairlie's punctures[2] $\sigma_i^{(F)} = \zeta_i$ always constitute one of them for any multiplicity n in four dimensions.

[1] Note that we are not writing the covariant integration volume for simplicity. However, the counting is exactly the same.

[2] We are emphasizing here the upper index in $\sigma_i^{(F)}$ for the sake of clarity. Nevertheless, we will be refering to Fairlie's punctures simply as σ_i in the rest of the chapter.

Four-point solution. There is one independent equation to solve, but no computations are needed because we already know about Fairlie's punctures. There is only **one** solution $\sigma^{(1)} = \sigma^{(F)}$, which in terms of the Sudakov parameters reads

$$\sigma_p^{(F)} = \infty \ , \quad \sigma_q^{(F)} = 0 \ , \quad \sigma_{p'}^{(F)} = -\frac{\hat{Q}_1}{\alpha} \ , \quad \sigma_{q'}^{(F)} = \frac{\hat{Q}_1}{1-\alpha} \ . \tag{4.2.9}$$

One might think, since the SE have real coefficients $s_{ij} \in \mathbb{R}$, that the complex conjugate $\left(\sigma^{(F)}\right)^*$ is a different solution of the equations. However, although it is indeed a solution, it turns out to be equivalent to $\sigma^{(F)}$. Considering the general $SL(2,\mathbb{C})$ transformation in Eq. (4.2.4) it is straightforward to check that

$$\sigma_i^{(F)} \mapsto \left(\sigma_i^{(F)}\right)^* \qquad \text{for} \qquad \frac{1}{|\hat{Q}_1|}\begin{pmatrix} \hat{Q}_1^* & 0 \\ 0 & \hat{Q}_1 \end{pmatrix} \in SL(2,\mathbb{C}) \ . \tag{4.2.10}$$

Remember that the kinematics of any 4-point process can be entirely described by a single parameter. Consequently, despite having used two different variables for the punctures —i.e. α and $\hat{Q}_1$—, there is only one degree of freedom characterizing the solution in Eq. (4.2.9), which is implicitly constrained by the onshellness of the external particles

$$p'^{\,2} = q'^{\,2} = 0 \quad \Rightarrow \quad |\hat{Q}_1|^2 = \alpha(1-\alpha) \ . \tag{4.2.11}$$

Five-point solutions. There are **two** different solutions, $\sigma^{(1)} = \sigma^{(F)}$ and $\sigma^{(2)} = \left(\sigma^{(F)}\right)^*$, stemming from two independent equations. In terms of Sudakov parameters, they can be expressed as

$$\sigma_p^{(F)} = \infty \ , \quad \sigma_q^{(F)} = 0 \ , \quad \sigma_{p'}^{(F)} = \frac{\hat{Q}_1}{\beta_1} \ , \quad \sigma_{q'}^{(F)} = \frac{\hat{Q}_2}{1+\beta_2} \ , \quad \sigma_k^{(F)} = \frac{\hat{Q}_1 - \hat{Q}_2}{\beta_1 - \beta_2} \ . \tag{4.2.12}$$

Analogously, one should bear in mind the implicit constraints derived from onshellness

$$p'^{\,2} = q'^{\,2} = k^2 = 0 \quad \Rightarrow \quad \left\{ \begin{array}{rcl} |\hat{Q}_1|^2 & = & \beta_1(\alpha_1 - 1) \ , \\ |\hat{Q}_1 - \hat{Q}_2|^2 & = & (\alpha_1 - \alpha_2)(\beta_1 - \beta_2) \ , \\ |\hat{Q}_2|^2 & = & \alpha_2(1+\beta_2) \ , \end{array} \right. \tag{4.2.13}$$

which translate into four degrees of freedom to describe the kinematics of a 5-point collision.

Notice that in both cases, only Fairlie's punctures are needed to write the SE solutions. For higher-point processes, on the contrary, this is not the case anymore. It is worth mentioning that both Fairlie's solution and its complex conjugate can be found in the literature under a different name. First, it was found in [66] that, in agreement with what we have, there always exist two rational solutions to the SE for all multiplicities. Written in terms of helicity spinors, they coincide with our punctures

$$\frac{\langle qi\rangle}{\langle pi\rangle} = \left(\sigma_i^{(F)}\right)^* , \qquad \frac{[qi]}{[pi]} = \sigma_i^{(F)} . \tag{4.2.14}$$

They were called respectively *holomorphic* and *anti-holomorphic* solutions. Futhermore, a few years later, it was proven in [86] that in fact these solutions have much more implications than just encoding the kinematics of the amplitude in spin-1 theories. Specifically, they were found to be the only contributions to MHV and $\overline{\text{MHV}}$ tree-level amplitudes, respectively, after integrating over all of them. Accordingly, they were given the more popular name of *MHV* and $\overline{MHV}$ *solutions*

$$\left(\sigma_i^{(F)}\right)^* = \frac{\langle qi\rangle}{\langle pi\rangle} \equiv \sigma_i^{(\mathrm{MHV})} , \qquad \sigma_i^{(F)} = \frac{[qi]}{[pi]} \equiv \sigma_i^{(\overline{\mathrm{MHV}})} . \tag{4.2.15}$$

We will further develop this point below.

4.3 Simple case: φ^3 scalar theory

4.3.1 The four-point case

The Sudakov representation provides a very convenient framework for the evaluation of scattering amplitudes in the CHY formalism, notably simplifying the computations. To illustrate this, we focus now on the calculation of the four-point ordered amplitude in a φ^3 scalar theory. According to the general prescription given in [63, 64], the amplitude can be written as the following integral supported on the solution to the SE

$$\begin{aligned}\mathcal{A}_4^{\varphi^3} &= \int \mathrm{d}z_{p'}\delta\left(\mathcal{S}_{p'}\right)\frac{z_{pq}^2 z_{qq'}^2 z_{q'p}^2}{\left(z_{pq}z_{qq'}z_{q'p'}z_{p'p}\right)^2} \\ &= \int \frac{\mathrm{d}z_{p'}}{(z_{p'}-\sigma_{q'})^2}\delta\left(\frac{s_{p'q}}{z_{p'}} - \frac{s_{p'q'}}{z_{p'}-\sigma_{q'}}\right) ,\end{aligned} \tag{4.3.1}$$

where all gauge generators are taken to be equal to one. Here we have partially fixed the $SL(2,\mathbb{C})$ invariance by setting $z_p \to \infty$ and $z_q \to 0$ while leaving the third one

$$z_{q'} \to \sigma_{q'} = \frac{Q_1}{(1-\alpha)\sqrt{s}} , \tag{4.3.2}$$

free. The integral defining the scattering amplitude has just one integration left over the position of the p' puncture. To carry out this integral, we notice that the argument of the delta function has a single root located at [see (4.2.9)]

$$z_{p'} = -\frac{Q_1}{\alpha\sqrt{s}} . \tag{4.3.3}$$

Evaluating the derivative of $S_{p'}$ with respect to the integration variable at the zero (4.3.3) gives

$$\mathcal{J} \equiv \left.\frac{\partial S_{p'}}{\partial z_{p'}}\right|_{z_{p'}=-\frac{Q_1}{\alpha\sqrt{s}}} = \frac{s^2\alpha^2(\alpha-1)}{Q_1^2} + \frac{s\alpha^2(1-\alpha)^2}{Q_1^2} = \frac{s^2\alpha^3(\alpha-1)}{Q_1^2} \,, \tag{4.3.4}$$

so we can simply write

$$\begin{aligned}\mathcal{A}_4^{\varphi^3} &= \int \mathrm{d}z_{p'} \left[z_{p'} - \frac{Q_1}{(1-\alpha)\sqrt{s}}\right]^{-2} \frac{Q_1^2}{s^2\alpha^3(\alpha-1)}\, \delta\left(z_{p'} + \frac{Q_1}{\alpha\sqrt{s}}\right) \\ &= \left[\frac{s\alpha^2(1-\alpha)^2}{Q_1^2}\right]\left[\frac{Q_1^2}{s^2\alpha^3(\alpha-1)}\right] = \frac{(\alpha-1)}{s\alpha} = \frac{1}{s} + \frac{1}{t} \,. \end{aligned} \tag{4.3.5}$$

Notice that the phase introduced in Q_1, which contains the azimuthal angle dependence, cancels out in the final expression for the amplitude. This is only natural, since θ_1 can be set to zero by using the residual $SL(2,\mathbb{C})$ transformations leaving invariant the position of the punctures associated with the incoming particles. Using this Sudakov parametrization, we see how the boundary of the 4-punctured sphere corresponding to the limit $\alpha \to 0$ is dominated by the $t = 0$ pole, while at the other branch of the boundary $\alpha \to 1$ the amplitude vanishes. At the equator $\alpha = \frac{1}{2}$ the amplitude is completely dominated by the pole at $s = 0$.

4.3.2 The five-point case

Having the two solutions to the SE, we are now ready to calculate the five-point amplitude for the φ^3 scalar theory. Using the same partial fixing of $SL(2,\mathbb{C})$ as in the calculation of the four-point amplitude in Eq. (4.3.1), we are left with the computation of the integral over the position of the punctures associated with p' and q', namely

$$\begin{aligned}\mathcal{A}_5^{\varphi^3} &= \int \mathrm{d}z_{p'}\mathrm{d}z_{q'}\, \delta\left(\mathcal{S}_{p'}\right)\delta\left(\mathcal{S}_{q'}\right) \frac{z_{pq}^2 z_{qk}^2 z_{kp}^2}{\left(z_{pq}z_{qq'}z_{q'k}z_{kp'}z_{p'p}\right)^2} \\ &= \int \mathrm{d}z_{p'}\mathrm{d}z_{q'}\, \delta\left(\mathcal{S}_{p'}\right)\delta\left(\mathcal{S}_{q'}\right) \frac{z_k^2}{z_{q'}^2 z_{q'k}^2 z_{kp'}^2} \,. \end{aligned} \tag{4.3.6}$$

To solve the delta function, we have to calculate the Jacobian

$$\mathcal{J} = \frac{\partial \mathcal{S}_{p'}}{\partial z_{p'}}\frac{\partial \mathcal{S}_{q'}}{\partial z_{q'}} - \frac{\partial \mathcal{S}_{p'}}{\partial z_{q'}}\frac{\partial \mathcal{S}_{q'}}{\partial z_{p'}} \,. \tag{4.3.7}$$

Things can be made simpler if we rewrite the SE associated to q' in the form

$$\frac{1}{s}\mathcal{S}_{q'} = \frac{\alpha_2}{z_{q'}} + \frac{(1-\alpha_1+\alpha_2-\beta_1+\beta_2)}{z_{p'q'}} + \frac{(\alpha_1+\beta_1)}{z_{kq'}} \tag{4.3.8}$$

$$= \frac{\big(\alpha_2 z_{p'} + (1-\alpha_1-\beta_1+\beta_2)z_{q'}\big)}{z_{p'q'}z_{kq'}}\left[\frac{z_k}{z_{q'}} - \frac{(\alpha_2-\alpha_1-\beta_1)\,z_{p'} + (1+\beta_2)z_{q'}}{\alpha_2 z_{p'} + (1-\alpha_1-\beta_1+\beta_2)z_{q'}}\right] .$$

What makes this expression useful is that we have isolated the zero due to Eq. (3.6.25). We can then write one of the derivatives on support of the SE as

$$\begin{aligned}\frac{1}{s}\frac{\partial \mathcal{S}_{q'}}{\partial z_{q'}}\bigg|_{\mathrm{SE}} &= \frac{\alpha_2\sigma_{p'} + (1-\alpha_1-\beta_1+\beta_2)\sigma_{q'}}{\sigma_{p'q'}\sigma_{kq'}}\frac{\partial}{\partial\sigma_{q'}}\left[\frac{\sigma_k}{\sigma_{q'}} - \frac{(\alpha_2-\alpha_1-\beta_1)\,\sigma_{p'} + (1+\beta_2)\sigma_{q'}}{\alpha_2\sigma_{p'} + (1-\alpha_1-\beta_1+\beta_2)\sigma_{q'}}\right] \\ &= \frac{(\alpha_1-\alpha_2-\beta_1)\alpha_2\sigma_{p'} + \big[(\alpha_1-\alpha_2)\beta_2 + (\alpha_2-1)\beta_1\big]\sigma_{q'}}{\alpha_2\sigma_{p'} + (1-\alpha_1)\sigma_{q'}}\left(\frac{\sigma_{p'}}{\sigma_{q'}\sigma_{p'q'}\sigma_{kq'}}\right) ,\end{aligned} \tag{4.3.9}$$

where we have used Eqs. (3.6.25) and (3.6.26). This can be further simplified by reintroducing σ_k to write

$$\frac{\partial \mathcal{S}_{q'}}{\partial z_{q'}}\bigg|_{\mathrm{SE}} = s\frac{(1+\beta_2)\sigma_{q'}^2 + \alpha_2\sigma_k\sigma_{p'}}{\sigma_{q'}^2\sigma_{p'q'}\sigma_{q'k}} . \tag{4.3.10}$$

We now repeat the same procedure for the SE associated to p', isolating the contribution to the zero

$$\frac{1}{s}\mathcal{S}_{p'} = \frac{(\alpha_2-\beta_1+\beta_2)z_{p'} + (1-\alpha_1)z_{q'}}{z_{p'q'}z_{p'k}}\left[\frac{z_k}{z_{p'}} + \frac{(\alpha_1-1-\alpha_2-\beta_2)z_{q'} + \beta_1 z_{p'}}{(\alpha_2-\beta_1+\beta_2)z_{p'} + (1-\alpha_1)z_{q'}}\right] , \tag{4.3.11}$$

and differentiating with respect to $z_{p'}$:

$$\frac{\partial \mathcal{S}_{p'}}{\partial z_{p'}}\bigg|_{\mathrm{SE}} = s\frac{\beta_1\sigma_{p'}^2 + (\alpha_1-1)\sigma_k\sigma_{q'}}{\sigma_{p'}^2\sigma_{p'q'}\sigma_{p'k}} . \tag{4.3.12}$$

Note that the two partial derivatives (4.3.10) and (4.3.12) can be mapped to each other by the replacements

$$\begin{aligned} p' &\longleftrightarrow q' , \\ -\beta_1 &\longleftrightarrow 1+\beta_2 , \\ \alpha_1 &\longleftrightarrow 1-\alpha_2 . \end{aligned} \tag{4.3.13}$$

Similar techniques allow to obtain the remaining two derivatives

$$\frac{\partial \mathcal{S}_{q'}}{\partial z_{p'}}\bigg|_{\mathrm{SE}} = s\frac{(\alpha_1+\beta_1)\,(1-\alpha_1+\alpha_2-\beta_1+\beta_2)}{(\beta_1-\beta_2)\sigma_k + (\alpha_1-\alpha_2)\sigma_{p'}}\left(\frac{\sigma_k}{\sigma_{p'q'}\sigma_{q'k}}\right) ,$$

$$\left.\frac{\partial \mathcal{S}_{p'}}{\partial z_{q'}}\right|_{\mathrm{SE}} = s\frac{(1-\alpha_1+\alpha_2-\beta_1+\beta_2)(\alpha_2+\beta_2)}{(\beta_1-\beta_2)\sigma_k+(\alpha_1-\alpha_2)\sigma_{q'}}\left(\frac{\sigma_k}{\sigma_{p'q'}\sigma_{p'k}}\right) , \tag{4.3.14}$$

which are also related by the transformations (4.3.13).

Returning to the calculation of the amplitude, we can finally write it as the sum over the two corresponding solutions as

$$\begin{aligned}\mathcal{A}_5^{\varphi^3} &= \int \mathrm{d}z_{p'}\mathrm{d}z_{q'}\, \mathcal{J}^{-1}\delta\big(z_{p'}-\sigma_{p'}\big)\delta\big(z_{q'}-\sigma_{q'}\big)\frac{z_k^2}{z_{q'}^2 z_{q'k}^2 z_{kp'}^2} + \text{c.c.}\\ &= \frac{2}{s^2}\mathrm{Re}\left[\left(\frac{\sigma_{p'q'}^2}{\sigma_{q'k}\sigma_{p'k}}\right)\frac{1}{\mathcal{L}-\mathcal{R}}\right] \\ &= \frac{1}{s^2}\left[\frac{1}{\alpha_1+\beta_1}-\frac{1}{\alpha_2+\beta_2}+\frac{1}{(\alpha_1+\beta_1)\beta_1}-\frac{1}{\beta_1\alpha_2}+\frac{1}{\alpha_2(\alpha_2+\beta_2)}\right] .\end{aligned} \tag{4.3.15}$$

Here we have used the notation

$$\mathcal{L} \equiv \left[\beta_1\frac{\sigma_{p'}}{\sigma_k}+(\alpha_1-1)\frac{\sigma_{q'}}{\sigma_{p'}}\right]\left[\alpha_2+(1+\beta_2)\frac{\sigma_{q'}^2}{\sigma_k\sigma_{p'}}\right] , \tag{4.3.16}$$

$$\mathcal{R} \equiv \frac{(1-\alpha_1+\alpha_2-\beta_1+\beta_2)^2\,(\alpha_1+\beta_1)\,(\alpha_2+\beta_2)\sigma_{q'}^2}{\big[(\beta_1-\beta_2)\sigma_k+(\alpha_1-\alpha_2)\sigma_{p'}\big]\big[(\beta_1-\beta_2)\sigma_k+(\alpha_1-\alpha_2)\sigma_{q'}\big]} , \tag{4.3.17}$$

as well as the explicit expressions for the positions of the punctures given in Eq. (4.2.12). The on-shell conditions (4.2.13) can be written in the form

$$2\cos(\theta_1-\theta_2) = \frac{\alpha_2-\beta_1+\alpha_1\beta_2+\alpha_2\beta_1}{\sqrt{(\alpha_1-1)\beta_1}\sqrt{(1+\beta_2)\alpha_2}} , \tag{4.3.18}$$

and turn out to be really useful for the numerical evaluation of our expressions and showing how the relative phase depends on the Sudakov variables.

It is also possible to write an alternative expression for the amplitude (4.3.15) as

$$\mathcal{A}_5^{\varphi^3} = \frac{2}{s^2}\mathrm{Re}\left[\left(\frac{\sigma_{p'}}{\sigma_{q'}}\right)\frac{1}{L\widetilde{L}-R\widetilde{R}}\right] , \tag{4.3.19}$$

where

$$\begin{aligned} L &= \frac{\sigma_{p'k}}{\sigma_{p'q'}}\left[(\alpha_1-1)\frac{\sigma_{q'}}{\sigma_{p'}}+\beta_1\frac{\sigma_{p'}}{\sigma_{q'}}\right] ,\\ R &= \left(\frac{\sigma_{p'}\sigma_{p'k}}{\sigma_{p'q'}}\right)\frac{(1-\alpha_1+\alpha_2-\beta_1+\beta_2)(\alpha_1+\beta_1)}{(\alpha_1-\alpha_2+\beta_1)\sigma_{p'}-(1+\beta_2)\sigma_{q'}} ,\end{aligned} \tag{4.3.20}$$

and the quantities with tilde are defined by implementing the replacements (4.3.13) in the form

$$\widetilde{\mathscr{O}}\Big(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1 - \theta_2\Big) = \mathscr{O}\Big(1 - \alpha_2, 1 - \alpha_1, -1 - \beta_2, -1 - \beta_1, \theta_2 - \theta_1\Big) \, . \tag{4.3.21}$$

The reason behind the existence of such a simple representation is the freedom to redefine the phase in the projective variables. This is part of the residual $SL(2, \mathbb{C})$ freedom present in our approach, after fixing the punctures associated with the two incoming particles. One interesting feature of this new compact representation in Eq. (4.3.19) is that it could be related to the mathematical properties of the zeros of the amplitude in an original way, deepening our understanding from the approach in Chapter 2.

4.4 Biadjoint scalar, Yang-Mills & Einstein-Hilbert gravity

In the last section we have been working directly with the integral representation of the CHY formalism so far

$$\mathcal{A}_n = \int \frac{\mathrm{d}^n z}{\mathrm{vol}[SL(2,\mathbb{C})]} \prod_{i=1}^{n}{}' \, \delta\left(\sum_{j=1 \neq i}^{n} \frac{s_{ij}}{z_i - z_j}\right) \mathcal{I}(z, p, \varepsilon) \, . \tag{4.4.1}$$

Nevertheless, the presence of the delta functions allows us to do some rearrangements. Normally, having a delta of an arbitrary function $\delta\left(\mathcal{S}_i(z)\right)$ implies obtaining the zeros of that function and performing a change of variables into $\delta(z_i)$. This is in essence the strategy followed in the previous section for the scalar amplitude. Therefore, knowing the solutions to the SE, it is possible to rewrite the whole amplitude as a sum over the inequivalent zeros of the SE

$$\mathcal{A}_n = \sum_{i=1}^{(n-3)!} J(\sigma^{(i)}, p) \, \mathcal{I}(\sigma^{(i)}, p, \varepsilon) \, , \tag{4.4.2}$$

where the jacobian is the price to pay from the change of variables $J(z, p) = |\partial \mathcal{S}_i(z)/\partial z_j|'$, and the integrand $\mathcal{I}(z, p, \varepsilon)$ remains unchanged from the integral representation. Of course, the jacobian has to be computed in a proper way, taking into account the $SL(2, \mathbb{C})$ symmetry involved in the integral measure.

In light of the correct $SL(2, \mathbb{C})$ scaling, there are many different building blocks that one can plug inside the integrand to describe a big set of theories. In the rest of the section, we show the construction of some of the most basic objects and their explicit expressions in Sudakov parametrization for $n = 4$ and $n = 5$. We will focus afterwards in the integrands of the biadjoint scalar theory, Yang-Mills and Einstein-Hilbert gravity, following the results from the previous chapters.

4.4.1 Building blocks

Jacobian factor: $J(z,p)$

This function is trivially, by construction of the formalism, the same for all theories. It is obtained by writing the jacobian matrix of the SE

$$\Phi = \frac{\partial \mathcal{S}_a(z)}{\partial z_b} = \begin{cases} \frac{s_{ab}}{(z_a-z_b)^2}, & a \neq b\ , \\ -\sum_{c\neq a} \frac{s_{ac}}{(z_a-z_c)^2}, & a = b\ , \end{cases} \tag{4.4.3}$$

and computing the determinant of the corresponding minor $|\Phi^{ijk}_{pqr}|$ after deleting rows $\{ijk\}$ and columns $\{pqr\}$. The operation is refered in the literature as *reduced determinant.* More explicitly, the *jacobian factor* is

$$J(z,p) = (\text{det'}\Phi)^{-1} \qquad \text{where} \qquad \text{det'}\Phi := \frac{(-1)^{i+j+k+p+q+r}|\Phi^{ijk}_{pqr}|}{(z_{ij}z_{jk}z_{ki})(z_{pq}z_{qr}z_{rp})}\ . \tag{4.4.4}$$

The choice of $\{ijk\}$ and $\{pqr\}$ fixes the redundancy appearing in the SE and in the punctures respectively with origin on the $SL(2,\mathbb{C})$ symmetry. Deleting different indices gives rise to distinct jacobian functions of the z_i's, although they all consistently lead to the same expression when evaluated on the support of the SE.

For example, one of the possibilities for the 4-point jacobian factor would be

$$\begin{aligned} J_4(z,p) &= \frac{z^2_{qp'}z_{p'q'}z_{q'q}z_{pq}z_{p'p}}{-|\Phi^{pqp'}_{qp'q'}|} = \frac{z_{pq}z_{pp'}z^2_{qp'}z^2_{pq'}z_{qq'}z_{p'q'}}{1-\alpha} \\ &\xrightarrow{\sigma_p\to\infty,\,\sigma_q\to 0} \quad \frac{z^2_{p'}z_{p'q'}z_{q'}}{\alpha-1}\times\sigma_p^4\ . \end{aligned} \tag{4.4.5}$$

Notice that we have identically performed the same partial fixing as in the previous chapter for clarity. The $\sigma_p\to\infty$ dependence should disappear once all factors of the amplitude are glued together. Evaluated at Fairlie's solution, it reads

$$J_4(\sigma,p) = -\frac{\hat{Q}_1^4}{(\alpha-1)^3\alpha^3}\times\sigma_p^4 = \frac{1}{(1-\alpha)\alpha}\times\sigma_p^4\ , \tag{4.4.6}$$

where we have made use of onshellness in Eq. (4.2.11) and rotational invariance of the amplitude by fixing $\theta_1\to 0$.

Likewise, for the 5-point jacobian factor, one of the most compact expressions corresponds to

$$J_5(z,p) = \frac{z_{pp'}z_{p'q'}z_{q'p}z_{qp'}z_{p'k}z_{kq}}{|\Phi^{pp'q'}_{qp'k}|} = \frac{z_{pp'}z_{p'q'}z_{q'p}z_{qp'}z_{p'k}z_{kq}}{\frac{\alpha_1+\beta_1}{z^2_{pq}z^2_{q'k}} - \frac{\alpha_2(\beta_1-\beta_2)}{z^2_{qq'}z^2_{pk}}}$$

$$\xrightarrow{\sigma_p\to\infty,\,\sigma_q\to 0} \quad \frac{z_{p'} z_{p'q'} z_{q'}^2 z_{p'k} z_{q'k}^2 z_k}{z_{q'}^2\,(\alpha_1+\beta_1) - z_{q'k}^2\,\alpha_2(\beta_1-\beta_2)} \times \sigma_p^4\,. \tag{4.4.7}$$

In this case, the evaluation over the SE solutions looks a bit more lengthy, although still quite simple thanks to the use of Sudakov variables and the freedom we have to play around with onshellness [see Eqs. (4.2.13) and (3.6.10)]

$$\begin{aligned}
J_5(\sigma^{(1)},p) &= \frac{\hat{Q}_1\hat{Q}_2\left(\hat{Q}_1-\hat{Q}_2\right)\left(\beta_2\hat{Q}_1-\beta_1\hat{Q}_2\right)\left((1+\beta_2)\hat{Q}_1-\beta_1\hat{Q}_2\right)}{\beta_1^3(\beta_1-\beta_2)^3(1+\beta_2)^3} \\
&\quad\times \frac{\left((1+\beta_2)\hat{Q}_1-(1+\beta_1)\hat{Q}_2\right)\times\sigma_p^4}{\left(\hat{Q}_1\hat{Q}_2^*-\hat{Q}_1^*\hat{Q}_2\right)}\,, \\
J_5(\sigma^{(2)},p) &= \frac{\hat{Q}_1^*\hat{Q}_2^*\left(\hat{Q}_1^*-\hat{Q}_2^*\right)\left(\beta_2\hat{Q}_1^*-\beta_1\hat{Q}_2^*\right)\left((1+\beta_2)\hat{Q}_1^*-\beta_1\hat{Q}_2^*\right)}{\beta_1^3(\beta_1-\beta_2)^3(1+\beta_2)^3} \\
&\quad\times \frac{\left((1+\beta_2)\hat{Q}_1^*-(1+\beta_1)\hat{Q}_2^*\right)\times\sigma_p^4}{\left(\hat{Q}_1^*\hat{Q}_2-\hat{Q}_1\hat{Q}_2^*\right)}\,.
\end{aligned} \tag{4.4.8}$$

Note from these expressions that the jacobian factor itself mimics the same behavior of the solutions, i.e.

$$\sigma^{(2)} = \left(\sigma^{(1)}\right)^* \quad \Rightarrow \quad J_5(\sigma^{(2)},p) = J_5(\sigma^{(1)},p)^*\,. \tag{4.4.9}$$

Parke-Taylor factor: $C(\rho,z)$

Similar to the Parke-Taylor formula for MHV color-ordered amplitudes in the spinor-helicity formalism, let us define the *Parke-Taylor factor* for CHY amplitudes as

$$C(\rho,z) = \frac{c_{\rho(1)\ldots\rho(n)}}{(z_{\rho(1)}-z_{\rho(2)})(z_{\rho(2)}-z_{\rho(3)})\ldots(z_{\rho(n)}-z_{\rho(1)})}\,, \tag{4.4.10}$$

where $c_{\rho(1)\ldots\rho(n)} = \mathrm{tr}(T^{a_{\rho(1)}}\ldots T^{a_{\rho(n)}})$ is the color trace matching the ordering ρ of the external legs. Full color stripped amplitudes can then make use of the following building block

$$\begin{aligned}
\sum_{\rho\in S_n/\mathbb{Z}_n} C(\rho,z) &= \sum_{\rho\in S_{n-1}} \frac{c_{1,\rho(2)\ldots\rho(n)}}{(z_1-z_{\rho(2)})(z_{\rho(2)}-z_{\rho(3)})\ldots(z_{\rho(n)}-z_1)} \\
&\xrightarrow{\sigma_p\to\infty} \quad -\sum_{\rho\in S_{n-1}} \frac{c_{1,\rho(2)\ldots\rho(n)}}{(z_{\rho(2)}-z_{\rho(3)})\ldots(z_{\rho(n-1)}-z_{\rho(n)})} \times \frac{1}{\sigma_p^2}\,,
\end{aligned} \tag{4.4.11}$$

with the sum running over non-cyclic permutations. Apparently, this factor is much simpler than the jacobian. Here we have an example for an arbitrary 4-point ordering and a generic color structure

$$C(\{1,2,4,3\},z) = \frac{c_{1243}}{z_{pq}z_{qq'}z_{q'p'}z_{p'p}} \xrightarrow{\sigma_p\to\infty,\sigma_q\to 0} \frac{-c_{1243}}{z_{q'}z_{p'q'}} \times \frac{1}{\sigma_p^2} \ . \tag{4.4.12}$$

Evaluated at Fairlie's solution, it takes the form

$$C(\{1,2,4,3\},\sigma) = c_{1243}(1-\alpha) \times \frac{1}{\sigma_p^2} \ . \tag{4.4.13}$$

Analogously, a 5-point Parke-Taylor factor example would be

$$C(\{1,2,4,5,3\},z) = \frac{c_{12453}}{z_{pq}z_{qq'}z_{q'k}z_{kp'}z_{p'p}} \xrightarrow{\sigma_p\to\infty,\sigma_q\to 0} \frac{-c_{12453}}{z_{q'}z_{q'k}z_{p'k}} \times \frac{1}{\sigma_p^2} \ . \tag{4.4.14}$$

This particular ordering, as in the 4-point case, has been chosen from the pure φ^3 scalar amplitudes obtained in Section 4.3, in order to compare later results. Evaluated at the SE solutions, it reads

$$\begin{aligned} C(\{1,2,4,5,3\},\sigma^{(1)}) &= \frac{-\beta_1(\beta_1-\beta_2)^2(1+\beta_2)^2\, c_{12453}}{\hat{Q}_2\left(\beta_2\hat{Q}_1-\beta_1\hat{Q}_2\right)\left((1+\beta_2)\hat{Q}_1-(1+\beta_1)\hat{Q}_2\right)} \times \frac{1}{\sigma_p^2} \ , \\ C(\{1,2,4,5,3\},\sigma^{(2)}) &= \frac{-\beta_1(\beta_1-\beta_2)^2(1+\beta_2)^2\, c_{12453}}{\hat{Q}_2^*\left(\beta_2\hat{Q}_1^*-\beta_1\hat{Q}_2^*\right)\left((1+\beta_2)\hat{Q}_1^*-(1+\beta_1)\hat{Q}_2^*\right)} \times \frac{1}{\sigma_p^2} \ . \end{aligned} \tag{4.4.15}$$

In this occasion, due to the presence of an arbitrary color factor, we have in general that $C(\rho,\sigma^{(1)}) \neq C(\rho,\sigma^{(2)})^*$. We could write instead

$$\frac{C(\rho,\sigma^{(1)})}{c_\rho} = \left(\frac{C(\rho,\sigma^{(2)})}{c_\rho}\right)^* \ . \tag{4.4.16}$$

Polarization factor: $E(z,p,\varepsilon)$

In order to come accross with alternative and more complex building blocks depending on momentum or even polarization vectors, one could first start thinking about the matricial form of the SE

$$M_A^{ab} = \begin{cases} \frac{2p_a\cdot p_b}{z_a-z_b} \ , & a\neq b \ , \\ 0 \ , & a=b \ . \end{cases} \tag{4.4.17}$$

It is easy to see that

$$\mathcal{S}_a(z) \equiv \sum_{b\neq a} \frac{s_{ab}}{z_a - z_b} = 0 \quad \Leftrightarrow \quad M_A \cdot \mathbf{1} = \mathbf{0} \,, \tag{4.4.18}$$

where $\mathbf{1} = (1, \overset{(n)}{\ldots}, 1)$ and $\mathbf{0} = (0, \overset{(n)}{\ldots}, 0)$. The most straightforward step to obtain a polynomial function on p^μ from this matrix is by means of its determinant, although notice again that the redundancies coming from the $SL(2,\mathbb{C})$ symmetry force to consider instead a reduced determinant det'M_A as in Eq. (4.4.4).

A direct generalization of this construction for the matrix M_A in Eq. (4.4.17) arises simply by including the polarization vectors into the following $n \times n$ matrices

$$M_B^{ab} = \begin{cases} \frac{2\varepsilon_a\cdot\varepsilon_b}{z_a - z_b}\,, & a \neq b\,, \\ 0\,, & a = b\,, \end{cases} \qquad M_C^{ab} = \begin{cases} \frac{2\varepsilon_a\cdot p_b}{z_a - z_b}\,, & a \neq b\,, \\ -\sum_{j\neq a} \frac{2\varepsilon_a\cdot p_j}{z_a - z_j}\,, & a = b\,, \end{cases} \tag{4.4.19}$$

and combining them into the $2n \times 2n$ antisymmetric matrix

$$\Psi = \begin{pmatrix} M_A & -M_C^T \\ M_C & M_B \end{pmatrix} \,. \tag{4.4.20}$$

The reduced determinant can be used also to obtain a polynomial function on p^μ and ε^μ. However, it brings us immediately into an expression which is bilinear in the polarization vectors i.e.

$$\text{det'}\Psi \sim (\ldots \varepsilon_i^\mu \varepsilon_i^\nu \ldots)\, \mathcal{T}_{\ldots\mu\nu\ldots} \,. \tag{4.4.21}$$

In essence, such an integrand would be pointing towards a potential candidate for a spin-2 theory; hopefully a well-defined gravity. We will see later that this is indeed the case.

The correct object for the construction of a spin-1 gauge theory comes from a well-known property of the determinant for even-dimensional matrices. This being the case for Ψ, the determinant can always be written as the square of a different polynomial function called the *pfaffian*. Trivially this new function is linear in the polarization vector, and it is needed to remove again the excess of information by defining a reduced version. Choosing the same rows and columns $\{ij\}$ for $1 \leq i < j \leq n$, we make the following definition

$$\text{Pf'}\Psi := \frac{(-1)^{i+j}}{2(z_i - z_j)}\, \text{Pf}\left[\Psi_{ij}^{ij}\right] \,, \tag{4.4.22}$$

which receives the name of *polarization factor* $E(z,p,\varepsilon) = \text{Pf'}\Psi$ in the formalism. The factor in front has the same meaning as in the reduced determinant, making the function to be $SL(2,\mathbb{C})$-covariant[3].

Some remark can be made in view of Eqs. (4.4.19), (4.4.20) and (4.4.22). Imagine that

[3]Notice that this time the choice of rows and columns must be the same for a consistent definition with the reduced determinant.

we shift any of the polarization vectors ε_i^μ by a multiple of the corresponding momentum p_i^μ —i.e. we perform a gauge transformation—. The whole $(i+n)$th-column and $(i+n)$th-row will be shifted proportionately by some term identical to the ith-column and ith-row themselves. Since we are computing the pfaffian[4] of the matrix, which remains unchanged, it means that the polarization factor is manifestly gauge invariant.

In the following, we are going to use the convention

$$\varepsilon_i^+ = \frac{1}{\sqrt{2}}\left(\zeta_i, 1, i, \zeta_i\right) , \qquad \varepsilon_i^- = \frac{1}{\sqrt{2}}\left(\bar{\zeta}_i, 1, -i, \bar{\zeta}_i\right) , \tag{4.4.23}$$

satisfying the usual conditions of vanishing modulus and transversality

$$\varepsilon_i^\pm \cdot \varepsilon_i^\pm = 0 = \varepsilon_i^\pm \cdot p_i , \qquad \varepsilon_i^\pm \cdot \varepsilon_i^\mp = -1 . \tag{4.4.24}$$

Working on the center-of-mass reference frame, and having incoming particles along the z-axis, it is clear that $|\zeta_p| \to \infty$ and $|\zeta_q| \to 0$. Therefore, taking advantage of gauge invariance in $E(z, p, \varepsilon)$, in order to avoid undesirable infinities, it will be useful to perform a gauge transformation to the first particle $\varepsilon_p^\pm \mapsto \varepsilon_p^\pm + \alpha^\pm p$. In particular, for $\alpha^+ = -\zeta_p\sqrt{2/s} = (\alpha^-)^*$ we have

$$\begin{aligned} \frac{1}{\sqrt{2}}\left(\zeta_p, 1, i, \zeta_p\right) \quad &\mapsto \quad \frac{1}{\sqrt{2}}\left(0, \frac{1-\zeta_p^2}{1+\zeta_p\bar{\zeta}_p}, i\,\frac{1+\zeta_p^2}{1+\zeta_p\bar{\zeta}_p}, \frac{2\,\zeta_p}{1+\zeta_p\bar{\zeta}_p}\right) , \\ \frac{1}{\sqrt{2}}\left(\bar{\zeta}_p, 1, -i, \bar{\zeta}_p\right) \quad &\mapsto \quad \frac{1}{\sqrt{2}}\left(0, \frac{1-\bar{\zeta}_p^2}{1+\zeta_p\bar{\zeta}_p}, -i\,\frac{1+\bar{\zeta}_p^2}{1+\zeta_p\bar{\zeta}_p}, \frac{2\,\bar{\zeta}_p}{1+\zeta_p\bar{\zeta}_p}\right) , \end{aligned}$$

which leads to $\varepsilon_p^\pm = (0, -1, \pm i, 0)/\sqrt{2}$ while keeping the same conditions in (4.4.24) untouched.

Having all the definitions, we are going to show now the explicit Sudakov dependencies for the 4- and 5-point polarization factors in the MHV configuration where negative helicities are assumed for the incoming particles. Specially in the 5-point case, it turns out to be really useful the Aitken's block diagonalization formula [109] to obtain the pfaffians and get affordable expressions in terms of computational effort.

The particular choice $\{pq\}$ leads to the following 4-point polarization factor

$$E_4(z, p, \mathrm{MHV}) = \frac{-\mathrm{Pf}\,[\Psi_{pq}^{pq}]}{2\,z_{pq}} = \frac{-2e^{2i\theta}}{z_{pp'}z_{qp'}z_{pq'}z_{qq'}} \xrightarrow{\sigma_p\to\infty,\sigma_q\to 0} \frac{-2}{z_{p'}z_{q'}} \times \frac{1}{\sigma_p^2} , \tag{4.4.25}$$

which evaluated over Fairlie's solution, transforms into

$$E_4(\sigma, p, \mathrm{MHV}) = \frac{-2\beta(1+\beta)}{|\hat{Q}_1|^2} \times \frac{1}{\sigma_p^2} = 2 \times \frac{1}{\sigma_p^2} . \tag{4.4.26}$$

[4]With analogous properties to those of the determinant.

In the 5-point case, the particular choice $\{q'k\}$ gives one of the shortest expressions

$$
\begin{aligned}
E_5(z,p,\mathrm{MHV}) &= \frac{-\mathrm{Pf}\left[\Psi^{q'k}_{q'k}\right]}{2\,z_{q'k}} \\
&\xrightarrow{\sigma_p\to\infty,\sigma_q\to 0} \frac{2\sqrt{2}\left(z_{p'k}\hat{Q}_1 - z_{q'k}\hat{Q}_2\right)}{\beta_1(\beta_1-\beta_2)(1+\beta_2)} \\
&\times \frac{z_{p'}z_{q'k}\hat{Q}_2\left(\beta_1\hat{Q}_2-\beta_2\hat{Q}_1\right) - z_{q'}z_{p'k}\hat{Q}_1\left((1+\beta_1)\hat{Q}_2-(1+\beta_2)\hat{Q}_1\right)}{z_{p'}z_{p'q'}z_{q'}z_{p'k}z_{q'k}z_k \times \sigma_p^2} \,.
\end{aligned}
\tag{4.4.27}
$$

Clearly, analogous to what happened in (4.4.15), the presence of polarization vectors will make $E_5(\sigma^{(1)},p,\mathrm{MHV}) \neq E_5(\sigma^{(2)},p,\mathrm{MHV})^*$. In particular, evaluated over the complex conjugate of Fairlie's solution $\sigma^{(2)}$, it reads

$$
\begin{aligned}
E_5(\sigma^{(2)},p,\mathrm{MHV}) &= \frac{2\sqrt{2}\beta_1^2(\beta_1-\beta_2)^2(1+\beta_2)^2}{\hat{Q}_1^*\hat{Q}_2^*\left(\hat{Q}_1^*-\hat{Q}_2^*\right)\left(\beta_2\hat{Q}_1^*-\beta_1\hat{Q}_2^*\right)\left((1+\beta_2)\hat{Q}_1^*-\beta_1\hat{Q}_2^*\right)} \\
&\times \frac{\left(\hat{Q}_1\hat{Q}_2^*-\hat{Q}_1^*\hat{Q}_2\right)}{\left((1+\beta_2)\hat{Q}_1^*-(1+\beta_1)\hat{Q}_2^*\right)} \times \frac{1}{\sigma_p^2}\,,
\end{aligned}
\tag{4.4.28}
$$

where we find some of the factors already encountered in the jacobian (4.4.8) and the Parke-Taylor (4.4.15) factor. On the contrary, evaluation over Fairlie's solution $\sigma^{(1)}$ gives

$$
E_5(\sigma^{(1)},p,\mathrm{MHV}) = 0\,. \tag{4.4.29}
$$

As expected from the characterization of the SE solutions in Eq. (4.2.15), it identically vanishes. Therefore, whenever computing a 5-point amplitude whose integrand contains the polarization factor $E(z,p,\varepsilon)$ in the MHV configuration, the sum over the two inequivalent solutions only has one contribution. The $\overline{\mathrm{MHV}}$ sector holds a similar structure. It is straightforward to check that

$$
\begin{aligned}
E_5(\sigma^{(1)},p,\mathrm{MHV}) &= E_5(\sigma^{(2)},p,\overline{\mathrm{MHV}}) = 0\,,\\
E_5(\sigma^{(2)},p,\mathrm{MHV}) &= E_5(\sigma^{(1)},p,\overline{\mathrm{MHV}})^*\,.
\end{aligned}
\tag{4.4.30}
$$

Basically this is the reason behind the naming of the two rational solutions of the SE as $\sigma^{(2)} \equiv \sigma^{(\mathrm{MHV})}$ and $\sigma^{(1)} \equiv \sigma^{(\overline{\mathrm{MHV}})}$. Moreover, the result is broader than that, becoming valid for any n-point amplitude in the MHV ($\overline{\mathrm{MHV}}$) sector, where every single term vanishes in the sum except the contribution coming from $\sigma^{(\mathrm{MHV})}$ ($\sigma^{(\overline{\mathrm{MHV}})}$).

It is worth showing at this point that, due to the presence of many common factors between the jacobian in Eq. (4.4.8) and the polarization factor in Eq. (4.4.28), some cancellations bring about the following simple expression

$$\begin{aligned} J_5(\sigma^{(2)}, p)\, E_5(\sigma^{(2)}, p, \text{MHV}) &= \frac{-2\sqrt{2}\,\sigma_p^2}{\beta_1(\beta_1-\beta_2)(1+\beta_2)} \\ &= 2\sqrt{2}\left(\prod_{i=1}^{5} \frac{1+\zeta_i\bar{\zeta}_i}{\omega_i}\right) > 0\,. \end{aligned} \tag{4.4.31}$$

4.4.2 Integrands & double copies: color-kinematics duality

Even though we have shown just a few basic building blocks, we already have enough ingredients to describe a wide variety of common and recognizable theories. Taking for example the Parke-Taylor factor and the SE matrix —both of them depending purely on momentum vectors—, we have the following scalar theories:

- *Special Galileon theory* [94, 110, 111]: $\mathcal{I}^{\text{sGal}} = (\det{}' M_A)^2$.
- *$U(N)$ non-linear Sigma model* [94, 112–114]: $\mathcal{I}^{\text{NLSM}} = C\,(\det{}' M_A)$.

Other closed formulas for spin-1 gauge theories also exist, in which one copy of the polarization factor is included

- Born-Infeld theory [115]: $\mathcal{I}^{\text{BI}} = \text{Pf}'\Psi\,(\text{Pf}' M_A)^2$.

More complex integrands with their corresponding theories and lagrangians can be found summarized in [116]. Nevertheless, in the rest of the section we are going to focus on the biadjoint scalar theory, Yang-Mills and Einstein-Hilbert gravity, being the simplest theories manifesting color-kinematics duality and for the sake of the discussion about planar radiation zeros started in Chapters 1 and 2.

The three integrands all together have the following structure:

$$\mathcal{I}^{\phi^3} = C(\rho, z)^2\,, \qquad \mathcal{I}^{\text{YM}} = C(\rho, z)\, E(z, p, \varepsilon)\,, \qquad \mathcal{I}^{\text{gravity}} = E(z, p, \varepsilon)^2\,. \tag{4.4.32}$$

In agreement with the linearity of the polarization factor in ε^μ, the spin of the three theories is in correspondence with the number of copies of the reduced pfaffian Pf'Ψ. Written in a more compact notation

$$\left.\begin{array}{cc} \text{Biadjoint scalar theory} & (\mathbf{s}=0) \\ \text{Yang-Mills theory} & (\mathbf{s}=1) \\ \text{Einstein-Hilbert gravity} & (\mathbf{s}=2) \end{array}\right\} \;\rightarrow\; \mathcal{I}^{(\mathbf{s})} = C(\rho, z)^{2-\mathbf{s}} E(z, p, \varepsilon)^{\mathbf{s}}\,. \tag{4.4.33}$$

Recall the original formulation of the color-kinematics duality [12, 15] where, starting from Yang-Mills amplitudes in a suitable decomposition, one can substitute the color factors by a second copy of the gauge numerators to obtain gravity; and the zeroth-copy prescription [45] where, having instead a second copy of the color factors allows to recover the corresponding biadjoint scalar amplitudes. We see that the integrand of the CHY formalism is naturally displaying the duality, getting rid of the additional constraints from the BCJ double-copy prescription and, consequently, reproducing it at a more fundamental level. The equivalence between the BCJ numerators and the expansion of the pfaffian in the formalism can be examined in [64, 117].

Let us see how the Sudakov representation is implemented in each of the theories.

Yang-Mills theory

Plugging together the Parke-Taylor factor in Eq. (4.4.13) and the polarization factor in Eq. (4.4.26) both with the jacobian in Eq. (4.4.6) we get

$$\mathcal{A}_4^{\mathrm{YM}}(1,2,4,3) = \frac{2}{\alpha}\, c_{1243} = \frac{2\, c_{1243}\, \langle pq\rangle^4}{\langle pq\rangle\langle qq'\rangle\langle q'p'\rangle\langle p'p\rangle}\,. \tag{4.4.34}$$

In a consistent way, the dependence on the puncture $\sigma_p \to \infty$ vanishes. We see that the 4-point amplitude gets a particularly simple expression, being just the inverse of the Sudakov variable α. The representation is even simpler than the Parke-Taylor formula of partial subamplitudes.

The 5-point amplitude, collecting factors from Eqs. (4.4.8), (4.4.15) and (4.4.28), becomes

$$\begin{aligned}\mathcal{A}_5^{\mathrm{YM}}(1,2,4,5,3) &= \frac{2\sqrt{2}(\beta_1-\beta_2)(1+\beta_2)\, c_{12453}}{\hat{Q}_2^*\left(\beta_2\hat{Q}_1^* - \beta_1\hat{Q}_2^*\right)\left((1+\beta_2)\hat{Q}_1^* - (1+\beta_1)\hat{Q}_2^*\right)} \\ &= \frac{2\sqrt{2}\, c_{12453}\, \langle pq\rangle^4}{\langle pq\rangle\langle qq'\rangle\langle q'k\rangle\langle kp'\rangle\langle p'p\rangle}\,. \end{aligned}\tag{4.4.35}$$

The only contribution coming from the complex conjugate of Fairlie's solution $\sigma^{(2)} \equiv (\sigma^{(F)})^* \equiv \sigma^{(\mathrm{MHV})}$ can be seen to coincide as well with the Parke-Taylor formula as expected. The discrepancy of global factors in both Eqs. (4.4.34) and (4.4.35) is due to the convention in the definition of the reduced pfaffian.

Einstein-Hilbert gravity

Replacing the Parke-Taylor factor $C(\rho, z)$ by a second copy of the polarization factor $E(z, p, \varepsilon)$ we get the following amplitudes. The 4-graviton amplitude, considering Eqs.(4.4.6) and (4.4.26), reads

$$\mathcal{M}_4^{\mathrm{gravity}} = \frac{4}{(1-\alpha)\alpha}\,, \tag{4.4.36}$$

whereas the 5-graviton amplitude, plugging together Eqs. (4.4.8) and (4.4.28), is of the form

$$\mathcal{M}_5^{\mathrm{gravity}} = \frac{8\,\beta_1(\beta_1-\beta_2)(1+\beta_2)}{\hat{Q}_1^*\hat{Q}_2^*\left(\hat{Q}_1^*-\hat{Q}_2^*\right)\left(\beta_2\hat{Q}_1^*-\beta_1\hat{Q}_2^*\right)\left((1+\beta_2)\hat{Q}_1^*-\beta_1\hat{Q}_2^*\right)}$$

$$\times \frac{\left(\hat{Q}_1^*\hat{Q}_2 - \hat{Q}_1\hat{Q}_2^*\right)}{\left((1+\beta_2)\hat{Q}_1^* - (1+\beta_1)\hat{Q}_2^*\right)} \,. \tag{4.4.37}$$

It might be interesting for other purposes to point out the fact that, before plugging the solutions of the SE into the integrand, each of the two copies of the pfaffian can be $SL(2,\mathbb{C})$-fixed independently. The double-copy structure can be found also in many other physical setups [12, 15, 17, 21, 93, 118] for which the CHY formalism could offer an alternative approach.

Biadjoint scalar ϕ^3 theory

Similarly, biadjoint scalar amplitudes can be obtained substituting the reduced pfaffian by a second copy of the gauge factor, thus implementing the zeroth copy prescription. From Eqs. (4.4.6) and (4.4.13) we see that the 4-scalar amplitude is

$$\mathcal{A}_4^{\phi^3}(1,2,4,3) = c_{1243}\bar{c}_{1243}\,\frac{1-\alpha}{\alpha} \,, \tag{4.4.38}$$

where, evidently, both color factors belong to a different family of generators. Analogously, from Eqs. (4.4.8) and (4.4.15) we get that the 5-scalar amplitude is

$$\begin{aligned}\mathcal{A}_5^{\phi^3}(1,2,4,5,3) &= c_{12453}\,\bar{c}_{12453} \\ &\times 2\,\mathrm{Re}\left[\frac{(\beta_1-\beta_2)(1+\beta_2)\,\hat{Q}_1\left(\hat{Q}_1-\hat{Q}_2\right)\left((1+\beta_2)\hat{Q}_1-\beta_1\hat{Q}_2\right)}{\beta_1\hat{Q}_2\left(\beta_2\hat{Q}_1-\beta_1\hat{Q}_2\right)\left(\hat{Q}_1\hat{Q}_2^*-\hat{Q}_1^*\hat{Q}_2\right)\left((1+\beta_2)\hat{Q}_1-(1+\beta_1)\hat{Q}_2\right)}\right] \,.\end{aligned} \tag{4.4.39}$$

The function 'Re[...]' arises when summing the two contributions to the amplitude and the fact that they are complex conjugate of each other according to Eqs. (4.4.9) and (4.4.16). The amplitude can be compared with the pure scalar case in Eq. (4.3.15) just by setting all color factors equal to one, $c_\rho\bar{c}_\rho \to 1$.

These *double-partial amplitudes*, using KLT orthogonality of the SE, can be related to the entries of the momentum kernel $S[\rho|\rho']$ in Eq. (3.6.36) in a very fascinating way [62, 64]

$$\sum_{i=1}^{(n-3)!} J(\sigma^{(i)},p)\frac{C(\rho,\sigma^{(i)})}{c_\rho}\frac{C(\rho',\sigma^{(i)})}{c_{\rho'}} = S^{-1}_{\mathrm{KLT}}[\rho|\rho'] \,, \tag{4.4.40}$$

with $\rho,\rho' \in S_n/\mathbb{Z}_n$ non-cyclic permutations related to the ordering of the external legs in the amplitude.

4.5 Gluon & graviton emission

Up to now we have seen the closed formulas for the biadjoint scalar theory, Yang-Mills theory and Einstein-Hilbert gravity, and how they are manifestly related in the CHY formalism through the color-kinematics duality. Nevertheless, this is not the only symmetry among the integrands present in the formalism. There are some powerful *operations* —e.g. *single-tracing*, *compactifying*, *squeezing*..., see [116] for a comprehensive review— that can be used to derive the formulas for many other theories, starting from the former three. A nice flow chart connecting various theories can be found in [94], where some examples are described for Einstein-Maxwell, Einstein-Yang-Mills or Dirac-Born-Infeld theories.

In this section we are going to use an operation called *compactification* to obtain the formulas in Sudakov representation of the gluon and graviton emission in the scattering of two scalars. For the building blocks[5], it is possible to make use of the construction of the polarization factor $E(z,p,\varepsilon)$. Considering some larger $D = 4+m$ dimensional space, we can redefine momentum and polarization vectors as

$$P_i^M = (p_i^\mu|\vec{0})\ , \qquad (\mathcal{E}_i^M)_{\text{gluon}} = (\varepsilon_i^\mu|\vec{0})\ , \qquad (\mathcal{E}_i^M)_{\text{scalar}} = (\vec{0}|e_i^I)\ , \tag{4.5.1}$$

where e_i^I is the standard orthonormal basis of the m-dimensional internal euclidean space and therefore the indices run as $M = 1,\ldots,D$; $\mu = 0,\ldots,3$ and $I = 1,\ldots,m$. These new dimensions can be understood as the space encoding some flavor index I_i on the i-th scalar i.e. $e_i^I = \delta^{I,I_i}$. One can see then that the vector products with respect to the new metric $\tilde{\eta}_{MN} = \text{diag}(1,-1,-1,-1|1,\overset{(m)}{\ldots},1)$, recover the desired behavior of the pfaffians for scalar particles

$$K_i\cdot\mathcal{E}_j = \begin{cases} k_i\cdot\varepsilon_j\ , & \text{for gluons}\ ,\\ 0\ , & \text{for scalars}\ ,\end{cases} \qquad \mathcal{E}_i\cdot\mathcal{E}_j = \begin{cases} \varepsilon_i\cdot\varepsilon_j\ , & \text{for gluons}\ ,\\ \delta^{I_i,I_j}\ , & \text{for scalars}\ ,\\ 0\ , & \text{else}\ .\end{cases} \tag{4.5.2}$$

Moreover, the vanishing of some of these vector products renders in Eq. (4.4.20) many of the Ψ matrix entries into zeros, allowing to write it in a block-diagonal form $\Psi = \text{diag}(\hat{\Psi}, X)$ and therefore to factorize the corresponding pfaffian. In particular we will have

$$E(z,p,\varepsilon) = \text{Pf}'[\hat{\Psi}]\text{Pf}[X]\ , \tag{4.5.3}$$

where the pfaffian Pf$[X]$ describes the scalar structure and only depends on the z_i's and the reduced pfaffian Pf'$[\hat{\Psi}]$ carries the momentum dependence and the polarization of the emitted gluon[6]

$$\text{Pf'}[\hat{\Psi}](z,p,\varepsilon^-) = \frac{-\text{Pf}[\hat{\Psi}^{q'k}_{q'k}]}{2z_{q'k}} = \frac{\beta_1\hat{Q}_2 z_{qp'}z_{pk} - \hat{Q}_1(\beta_2 z_{pp'}z_{qk} - \beta_1 z_{pq}z_{p'k})}{\sqrt{2}\,(\beta_1-\beta_2)\,z_{pq}z_{pp'}z_{qk}z_{p'k}z_{q'k}}$$

[5] Here we are considering the 5-point case —i.e. gluon emission from scattering of two scalars.
[6] All shown results correspond to negative helicity ε^- for the emitted gluon.

$$\xrightarrow{\sigma_p\to\infty,\sigma_q\to 0} \frac{\beta_1\hat{Q}_2 z_{p'} - \hat{Q}_1(\beta_1 z_{p'} - (\beta_1-\beta_2)z_k)}{\sqrt{2}\,(\beta_2-\beta_2)\, z_{p'k}z_k z_{q'k}\times\sigma_p} \,. \tag{4.5.4}$$

Evaluated on the SE solutions, it reads

$$\text{Pf'}[\hat{\Psi}](\sigma^{(1)}, p, \varepsilon^-) = 0 \,, \tag{4.5.5}$$

$$\text{Pf'}[\hat{\Psi}](\sigma^{(2)}, p, \varepsilon^-) = \frac{\beta_1(\beta_1-\beta_2)^2(1+\beta_2)\left(\hat{Q}_1\hat{Q}_2^* - \hat{Q}_1^*\hat{Q}_2\right)}{\sqrt{2}\left(\hat{Q}_2^*-\hat{Q}_1^*\right)\left(\beta_1\hat{Q}_2^*-\beta_2\hat{Q}_1^*\right)\left((1+\beta_1)\hat{Q}_2^*-(1+\beta_2)\hat{Q}_1^*\right)\times\sigma_p} \,. \tag{4.5.6}$$

There is no contribution from Fairlie's solution to this helicity configuration for the emitted gluon, similar to what we found for MHV Yang-Mills amplitudes in Eq. (4.4.30). In an analogous way, it is easy to see that

$$\begin{aligned} \text{Pf'}[\hat{\Psi}](\sigma^{(1)}, p, \varepsilon^-) &= \text{Pf'}[\hat{\Psi}](\sigma^{(2)}, p, \varepsilon^+) = 0 \,, \\ \text{Pf'}[\hat{\Psi}](\sigma^{(2)}, p, \varepsilon^-) &= \text{Pf'}[\hat{\Psi}](\sigma^{(1)}, p, \varepsilon^+)^* \,. \end{aligned} \tag{4.5.7}$$

It is also worth noting that both expressions (4.4.28) and (4.5.6) contain a $\hat{Q}_1\hat{Q}_2^* - \hat{Q}_1^*\hat{Q}_2$ factor in the numerator. This will be important when studying the planar limit because of its angular dependence

$$\hat{Q}_1\hat{Q}_2^* - \hat{Q}_1^*\hat{Q}_2 = 2i\, q_1^\perp q_2^\perp \sin(\theta_1-\theta_2) \,. \tag{4.5.8}$$

A distinction has to be made now for the second block of the $\Psi = \text{diag}(\hat{\Psi}, X)$ matrix depending on whether the scalars are considered to be distinguishable or indistinguishable particles:

Indistinguishable scalars. Implementation of indistinguishability is easy since we can just set $m=1$, meaning that all scalars carry the same flavor index. In this case

$$\text{Pf}[X]_{\text{indist}} = 4\left(\frac{1}{z_{pq'}z_{qp'}} + \frac{1}{z_{pq}z_{p'q'}} - \frac{1}{z_{pp'}z_{qq'}}\right) \xrightarrow{\sigma_p\to\infty,\sigma_q\to 0} \frac{4}{\sigma_p}\left(\frac{-1}{z_{p'}} + \frac{1}{z_{p'q'}} + \frac{1}{z_{q'}}\right) \,. \tag{4.5.9}$$

Evaluated over the SE solutions, it reads

$$\begin{aligned} \text{Pf}[X]_{\text{indist}}(\sigma^{(1)}, p) &= -\frac{4}{\sigma_p}\left(\frac{\beta_1}{\hat{Q}_1} + \frac{\beta_1(1+\beta_2)}{\beta_1\hat{Q}_2-(1+\beta_2)\hat{Q}_1} - \frac{1+\beta_2}{\hat{Q}_2}\right) \,, \\ \text{Pf}[X]_{\text{indist}}(\sigma^{(2)}, p) &= \left(\text{Pf}[X]_{\text{indist}}(\sigma^{(1)}, p)\right)^* \,. \end{aligned} \tag{4.5.10}$$

Distinguishable scalars. This would correspond to the presence of two flavor indices $m = 2$, accounting for the scattering of two distinct scalars. The pfaffian is

$$\mathrm{Pf}[X]_{\mathrm{dist}} = \frac{-4}{z_{pp'} z_{qq'}} \xrightarrow{\sigma_p \to \infty, \sigma_q \to 0} \frac{4}{z_{q'} \times \sigma_p} \, , \tag{4.5.11}$$

which, evaluated over the SE solutions, becomes

$$\mathrm{Pf}[X]_{\mathrm{dist}}(\sigma^{(1)}, p) = \frac{4(1+\beta_2)}{\hat{Q}_2 \times \sigma_p} \, , \qquad \mathrm{Pf}[X]_{\mathrm{dist}}(\sigma^{(2)}, p) = \frac{4(1+\beta_2)}{\hat{Q}_2^* \times \sigma_p} \, . \tag{4.5.12}$$

Incidentally, this factor is found to be one of the 'channels' inside equation (4.5.9).

4.5.1 Gluon emission

After compactifying the structure of the reduced pfaffian inside the polarization factor in Eq. (4.5.3), we get the following integrand for the gluon emission

$$\mathcal{I}^{\phi\bar{\phi}g} = C \left(\mathrm{Pf'}[\hat{\Psi}] \mathrm{Pf}[X] \right) \, . \tag{4.5.13}$$

Taking the single contribution from the $\sigma^{(2)}$ solution of the SE, we have that the corresponding partial amplitude in terms of Sudakov variables is

$$\mathcal{A}^{\phi\bar{\phi}g}\Big|_{\mathrm{dist}} = - \frac{4\,(\beta_1 - \beta_2)\hat{Q}_1^* \left((1+\beta_2)\hat{Q}_1^* - \beta_1 \hat{Q}_2^* \right)}{\sqrt{2}\,\beta_1 \left(\beta_2 \hat{Q}_1^* - \beta_1 \hat{Q}_2^* \right) \left((1+\beta_2)\hat{Q}_1^* - (1+\beta_1)\hat{Q}_2^* \right)} \frac{(1+\beta_2)}{\hat{Q}_2^*} \times c_{12453} \, , \tag{4.5.14}$$

$$\begin{aligned} \mathcal{A}^{\phi\phi g}\Big|_{\mathrm{indist}} &= \frac{4(\beta_1 - \beta_2)\hat{Q}_1^* \left((1+\beta_2)\hat{Q}_1^* - \beta_1 \hat{Q}_2^* \right)}{\sqrt{2}\,\beta_1 \left(\beta_2 \hat{Q}_1^* - \beta_1 \hat{Q}_2^* \right) \left((1+\beta_2)\hat{Q}_1^* - (1+\beta_1)\hat{Q}_2^* \right)} \\ &\quad \times \left(\frac{\beta_1}{\hat{Q}_1^*} + \frac{\beta_1(1+\beta_2)}{\beta_1 \hat{Q}_2^* - (1+\beta_2)\hat{Q}_1^*} - \frac{1+\beta_2}{\hat{Q}_2^*} \right) \times c_{12453} \, . \end{aligned} \tag{4.5.15}$$

Notice that, although it cannot be compared directly to the formulas in Eq. (2.3.2) due to the different color structure of the scalars, we have introduced the expression for the gluon emission in the scattering of distinguishable scalars with the purpose of performing the double-copy and generating the graviton emission amplitude.

4.5.2 Graviton emission

The integrand, according to color-kinematics duality, is

$$\mathcal{I}^{\Phi\bar{\Phi}G} = \left(\mathrm{Pf'}[\hat{\Psi}] \mathrm{Pf}[X] \right)^2 \, . \tag{4.5.16}$$

The partial ordering, as it was the case in Eq. (4.4.37), is not present anymore. Expressed in terms of Sudakov variables, we get

$$\mathcal{A}^{\Phi\bar{\Phi}G}\Big|_{\text{dist}} = \frac{8(\beta_1-\beta_2)(1+\beta_2)\hat{Q}_1^*\left((1+\beta_2)\hat{Q}_1^*-\beta_1\hat{Q}_2^*\right)\left(\hat{Q}_1^*\hat{Q}_2-\hat{Q}_1\hat{Q}_2^*\right)}{\beta_1\hat{Q}_2^*\left(\hat{Q}_1^*-\hat{Q}_2^*\right)\left(\beta_2\hat{Q}_1^*-\beta_1\hat{Q}_2^*\right)\left((1+\beta_2)\hat{Q}_1^*-(1+\beta_1)\hat{Q}_2^*\right)} \ . \qquad (4.5.17)$$

It is important to point out that, although coming from color-kinematics duality, this amplitude does not coincide with the putative gravitational amplitude obtained from the double-copy of the gauge amplitude in Eq. (2.3.2). Actually, such strategy is highly non-trivial whenever arbitrary color generators are considered. Some examples of double-copying in other scenarios can be found in [117] for the *non-linear sigma model* or in [93], where massive scalars are included in the theory.

4.6 Planar radiation zeros

Having displayed all the analytic structure of the different integrands, we are on the right track to understand the nature of the planar radiation zeros studied in Chapters 1 and 2 from the CHY formalism perspective. The main results center around 5-point tree-level amplitudes. On one side, Chapter 1 showed that Yang-Mills 5-gluon planar zeros are characterized by a color-dependent algebraic curve inside the projective plane spanned by the three stereographic coordinates labelling the direction of the outgoing momenta. On the other side, taking advantage of the relation between gluon and graviton amplitudes through the color-kinematics duality, it was shown that Einstein-Hilbert 5-graviton amplitudes vanish whenever the collision takes place in the planar configuration, without imposing any further kinematic constraints. Moreover, in Chapter 2 it was studied how these results remain invariant even when considering scalar particles as part of the matter content of the corresponding theories.

Let us see how the CHY formalism reproduces and sheds some light upon this behavior. First of all, this section gives explicit expressions for the planar limit of the building blocks presented in Section 4.4.1. From there, the origin of planar zeros is studied in detail for the different integrands in Section 4.4.2. Gluon and graviton emission are studied as well as a special case of the former.

4.6.1 Planar limit

In Sudakov parametrization, the 'planar dependence' enters via the azimuthal angles θ_i inside the variables $Q_i \equiv q_i^\perp e^{i\theta_i}$ encoding transverse momenta. In the 5-point case[7], there are two of such variables —i.e. $\hat{Q}_1$ and $\hat{Q}_2$—. The limit can be computed for instance by fixing the overall orientation of the process for one of the azimuthal angles and performing an appropiate Taylor expansion for the other e.g.

$$\theta_1 = 0, \pi \ , \qquad \theta_2 \equiv \epsilon \to 0 \ ; \qquad (4.6.1)$$

[7]The 4-point case is trivial since every collision is already planar in the center-of-mass reference frame.

although some subtleties have to be taken into account. From the on-shell conditions in Eq. (3.6.9) and total momentum conservation, we know that four independent variables are enough to describe the whole process, which we chose to be α_1, α_2, β_1 and β_2 in Chapter 3. However, in order to obtain more compact expressions, we decided to keep track of the $\hat{Q}_i$ variables during the computations, introducing some redundant parameters in the description and an ambiguous way of taking the proper planar limit. Eq. (4.6.1) becomes thus insufficient. For instance, from the identity in Eq (3.6.10) we have that

$$\lim_{\theta_1\to 0,\pi}\left(\lim_{\theta_2\to 0}[\alpha_2-\beta_1+\alpha_1\beta_2+\alpha_2\beta_1]\right)=2\eta\,\sqrt{\beta_1(\alpha_1-1)}\sqrt{\alpha_2(1+\beta_2)}\;, \tag{4.6.2}$$

although there is no explicit azimuthal angle dependence inside the limit. Notice that $\eta=\pm 1$ has been introduced as $\theta_1-\theta_2\to 0,\pi$ respectively, accounting for the two possible configurations. Therefore, after a careful Taylor expansion and some numerical tests we are led to the following results:

Jacobian factor: $J(\sigma^{(i)},p)\big|_{\mathbf{planar}}$**.** The planar expansion of Eq. (4.4.8) is

$$\begin{aligned}
J(\sigma^{(2)},p)\Big|_{\text{planar}} &\approx -\frac{\left(q_2^\perp-\eta q_1^\perp\right)\left(\beta_1 q_2^\perp-\eta\beta_2 q_1^\perp\right)\left(\beta_1 q_2^\perp-\eta(1+\beta_2)q_1^\perp\right)\times\sigma_p^4}{2\beta_1^3(\beta_1-\beta_2)^3(1+\beta_2)^3\left((1+\beta_1)q_2^\perp-\eta(1+\beta_2)q_1^\perp\right)^{-1}}\\
&\quad\times\left[\frac{i}{\epsilon}+\left(\frac{q_2^\perp}{q_2^\perp-\eta q_1^\perp}+\frac{\beta_1 q_2^\perp}{\beta_1 q_2^\perp-\eta\beta_2 q_1^\perp}+\frac{\beta_1 q_2^\perp}{\beta_1 q_2^\perp-\eta(1+\beta_2)q_1^\perp}\right.\right.\\
&\quad\left.\left.+\;\frac{(1+\beta_1)q_2^\perp}{(1+\beta_1)q_2^\perp-\eta(1+\beta_2)q_1^\perp}-1\right)\right]\;,\\
J(\sigma^{(1)},p)\Big|_{\text{planar}} &= \left(J(\sigma^{(2)},p)\Big|_{\text{planar}}\right)^{*}\;.
\end{aligned} \tag{4.6.3}$$

We can see that the jacobian diverges in the planar limit and that it is purely imaginary at leading order. The singularity comes from the factor $\hat{Q}_1\hat{Q}_2^*-\hat{Q}_1^*\hat{Q}_2$ in the denominator i.e.

$$\hat{Q}_1\hat{Q}_2^*-\hat{Q}_1^*\hat{Q}_2=2i\,q_1^\perp q_2^\perp\sin(\theta_1-\theta_2)\xrightarrow{\text{planar}}-2i\,q_1^\perp q_2^\perp\eta\,\epsilon\;. \tag{4.6.4}$$

Since scattering amplitudes are in general finite in this regime, it means that cancellations will occur when plugging it into the integrand. For example, we already saw that the polarization factor $E(\sigma^{(2)},p,\text{MHV})$ in Eq. (4.4.28) carries the same factor in its numerator.

Parke-Taylor factor: $C(\rho,\sigma^{(i)})\big|_{\mathbf{planar}}$**.** The planar expansion of Eq. (4.4.15) is

$$C(\{1,2,4,5,3\},\sigma^{(2)})\Big|_{\text{planar}}\approx\frac{-\beta_1(\beta_1-\beta_2)^2(1+\beta_2)^2\,c_{12453}}{q_2^\perp\left(\beta_1 q_2^\perp-\eta\beta_2 q_1^\perp\right)\left((1+\beta_1)q_2^\perp-\eta(1+\beta_2)q_1^\perp\right)\times\sigma_p^2}$$

$$
\begin{aligned}
&\times \left[1 + i\epsilon \left(\frac{\beta_1 q_2^\perp}{\beta_1 q_2^\perp - \eta \beta_2 q_1^\perp} \right.\right. \\
&\left.\left. + \; \frac{(1+\beta_1) q_2^\perp}{(1+\beta_1) q_2^\perp - \eta (1+\beta_2) q_1^\perp} + 1 \right)\right] \,, \\
\frac{C(\{1,2,4,5,3\},\sigma^{(1)})\Big|_{\text{planar}}}{c_{12453}} &= \left(\frac{C(\{1,2,4,5,3\},\sigma^{(2)})\Big|_{\text{planar}}}{c_{12453}} \right)^* \,. \qquad (4.6.5)
\end{aligned}
$$

The leading behavior is the same for different partial orderings. In this case, the factor becomes real[8] at leading order, without affecting the jacobian singularity for the whole amplitude. In the next section we will show how the biadjoint scalar theory deals with this issue concerning finite amplitudes.

Polarization factor: $E(\sigma^{(i)}, p, \varepsilon)\Big|_{\textbf{planar}}$**.** The planar expansion of Eq. (4.4.28) is

$$
\begin{aligned}
E(\sigma^{(2)}, p, \text{MHV})\Big|_{\text{planar}} &\approx \frac{4\sqrt{2}\beta_1^2(\beta_1-\beta_2)^2(1+\beta_2)^2 \left((1+\beta_1)q_2^\perp - \eta(1+\beta_2)q_1^\perp\right)^{-1}}{\left(q_2^\perp - \eta q_1^\perp\right)\left(\beta_1 q_2^\perp - \eta\beta_2 q_1^\perp\right)\left(\beta_1 q_2^\perp - \eta(1+\beta_2) q_1^\perp\right) \times \sigma_p^2} \\
&\times \left[-i\epsilon + \epsilon^2 \left(5 + \frac{\eta q_1^\perp}{q_2^\perp - \eta q_1^\perp} + \frac{\eta \beta_2 q_1^\perp}{\beta_1 q_2^\perp - \eta\beta_2 q_1^\perp} \right.\right. \\
&\left.\left. + \; \frac{\eta(1+\beta_2) q_1^\perp}{\beta_1 q_2^\perp - \eta(1+\beta_2) q_1^\perp} + \frac{\eta(1+\beta_2) q_1^\perp}{(1+\beta_1) q_2^\perp - \eta(1+\beta_2) q_1^\perp} \right)\right] \,, \\
E(\sigma^{(1)}, p, \overline{\text{MHV}})\Big|_{\text{planar}} &= \left(E(\sigma^{(2)}, p, \text{MHV})\Big|_{\text{planar}} \right)^* \,. \qquad (4.6.6)
\end{aligned}
$$

Recall that the evaluation of the polarization factor $E(z, p, \text{MHV})$ over the Fairlie solution $\sigma^{(1)} \equiv \sigma^{(F)}$ for the MHV sector identically vanishes. The behavior for the second solution $\sigma^{(2)}$ is remarkable since it also vanishes although at leading order, as expected from the limit in Eq. (4.6.4). This is the zero that eventually will compensate for the jacobian singularity, leading to a finite amplitude at least in Yang-Mills theory.

In summary, the leading behaviour of the different building blocks in the planar limit is

$$
\begin{aligned}
J(\sigma^{(i)}, p)\Big|_{\text{planar}} &\sim \mathcal{O}(1) + i\,\mathcal{O}\left(\tfrac{1}{\epsilon}\right) \sim i\,\mathcal{O}\left(\tfrac{1}{\epsilon}\right) \,, \\
\frac{C(\rho,\sigma^{(i)})}{c_\rho}\Big|_{\text{planar}} &\sim \mathcal{O}(1) + i\,\mathcal{O}(\epsilon) \sim \mathcal{O}(1) \,, \qquad (4.6.7) \\
E(\sigma^{(i)}, p, \varepsilon)\Big|_{\text{planar}} &\sim \mathcal{O}\left(\epsilon^2\right) + i\,\mathcal{O}(\epsilon) \sim i\,\mathcal{O}(\epsilon) \,.
\end{aligned}
$$

[8] Subjected to convention for the color structure constants.

4.6.2 Einstein-Hilbert gravity planar zeros

As it was already seen from the color-kinematics duality in Eq. (4.4.32), the Einstein-Hilbert integrand is built from two copies of the polarization factor. It means that the integrand contributes with a double zero to the amplitude in the planar limit

$$\mathcal{I}^{\text{gravity}}\Big|_{\text{planar}} = E(\sigma^{(2)}, p, \text{MHV})^2\Big|_{\text{planar}} \sim \mathcal{O}(\epsilon^2) \ . \tag{4.6.8}$$

Together with the single pole of the jacobian, this regime leads to a vanishing amplitude for any kinematic configuration

$$\mathcal{M}_5^{\text{gravity}}\Big|_{\text{planar}} \sim i\mathcal{O}(\epsilon) \ . \tag{4.6.9}$$

4.6.3 Yang-Mills planar zeros

In Yang-Mills theory, the cancellation between the single pole in the jacobian and the single zero of the integrand follows from

$$\mathcal{I}^{\text{YM}}\Big|_{\text{planar}} = C(\rho, \sigma^{(2)}) E(\sigma^{(2)}, p, \text{MHV})\Big|_{\text{planar}} \sim i\mathcal{O}(\epsilon) \ , \tag{4.6.10}$$

granting a finite amplitude

$$\mathcal{A}_5^{\text{YM}}\Big|_{\text{planar}} \sim \mathcal{O}(1) \ . \tag{4.6.11}$$

We can see from the global factor in Eqs. (4.6.3) and (4.6.6) that planar zeros cannot be characterized neither from the jacobian nor the polarization factor since they almost perfectly cancel each other [see Eq. (4.4.31) for concreteness]. Hence, according to Eq. (4.4.11), we have that planar zeros $\mathcal{A}_5^{\text{YM}}\big|_{\text{planar}} = 0$ are defined from the following constraint over the Parke-Taylor factor

$$\sum_{\rho \in S_{n-2}} C(\rho, \sigma^{(2)})\Big|_{\text{planar}} = 0 \quad \Leftrightarrow \quad \sum_{\rho \in S_3} \frac{c_{1,\rho(3),\rho(4),\rho(5),2}}{(z_{\rho(3)} - z_{\rho(4)})(z_{\rho(4)} - z_{\rho(5)}) z_{\rho(5)}}\Bigg|_{z \to \sigma^{(2)} \in \mathbb{R}} = 0 \ . \tag{4.6.12}$$

It might seem that this is a different condition compared to the projective curve shown in Chapter 1. However, after multiplying by $\prod_{i<j}(z_i - z_j) \neq 0$, it turns out that the zeros are given by

$$\sum_{\rho \in S_3} c_{1,\rho(3),\rho(4),\rho(5),2} \times \frac{\text{sgn}(\rho)\,(z_{\rho(3)} - z_{\rho(5)})}{z_{\rho(5)}}\Bigg|_{z_i \to \sigma_i^{(2)} = \zeta_i \in \mathbb{R}} = 0 \ , \tag{4.6.13}$$

which exactly reproduces[9] the result previously obtained in (1.3.4)

$$c_{15432}\frac{\zeta_k-\zeta_{p'}}{\zeta_{p'}}+c_{14352}\frac{\zeta_{q'}-\zeta_k}{\zeta_k}-c_{13452}\frac{\zeta_{p'}-\zeta_k}{\zeta_k}$$
$$+c_{13542}\frac{\zeta_{p'}-\zeta_{q'}}{\zeta_{q'}}-c_{15342}\frac{\zeta_k-\zeta_{q'}}{\zeta_{q'}}-c_{14532}\frac{\zeta_{q'}-\zeta_{p'}}{\zeta_{p'}}=0\,. \tag{4.6.14}$$

Non-planar radiation zeros. Inspecting the zero condition in Eq. (4.6.12), one might think that non-planar zeros are characterized as well by the same equation; and it is indeed the case. In the non-planar regime, radiation zeros would be defined by the two simultaneous conditions

$$\mathrm{Re}\left[\sum_{\rho\in S_{n-2}}C(\rho,\sigma^{(2)})\right]=0\,,\qquad \mathrm{Im}\left[\sum_{\rho\in S_{n-2}}C(\rho,\sigma^{(2)})\right]=0\,. \tag{4.6.15}$$

An scan over phase space shows that this definition restricts their localization to just a few isolated points in the physical region for some specific color configurations, as expected from the discussion in [33].

4.6.4 Biadjoint scalar planar zeros

In the pure scalar theory, with two distinct copies of the Parke-Taylor factor in the integrand, the single pole coming from the jacobian does not cancel directly anymore i.e.

$$\mathcal{I}^{\phi^3}\Big|_{\text{planar}}=C(\rho,\sigma^{(i)})^2\Big|_{\text{planar}}\sim\mathcal{O}(1)\quad\text{for}\quad i=1,2\,. \tag{4.6.16}$$

Instead, the non-presence of the polarization factor brings back the contribution from Fairlie's solution $\sigma^{(F)}\equiv\sigma^{(1)}$ to the whole amplitude, allowing the divergence to disappear

$$\mathcal{A}_5^{\phi^3}\Big|_{\text{planar}}\sim\mathrm{Re}\left[\mathcal{O}(1)+i\mathcal{O}\left(\frac{1}{\epsilon}\right)\right]\sim\mathcal{O}(1)\,. \tag{4.6.17}$$

This is possible due to the fact that both for the jacobian and the Parke-Taylor factor, SE solutions' contributions are complex conjugate of each other [see Eqs. (4.6.3) and (4.6.5)]

$$J(\sigma^{(1)},p)\Big|_{\text{planar}}=\left(J(\sigma^{(2)},p)\Big|_{\text{planar}}\right)^*\,,$$
$$\frac{C(\{1,2,4,5,3\},\sigma^{(1)})\Big|_{\text{planar}}}{c_{12453}}=\left(\frac{C(\{1,2,4,5,3\},\sigma^{(2)})\Big|_{\text{planar}}}{c_{12453}}\right)^*\,.$$

[9] Bear in mind the change of notation with respect to Chapters 1 and 2, i.e. $\{12345\}\equiv\{pqp'q'k\}$.

The particular planar zero condition for the 5-scalar amplitude in the CHY representation is not shown here, but it can be seen to be equivalent to the one obtained from Eq.(2.2.8). It is written explicitly in Appendix B.1.

4.6.5 Gluon & graviton emission planar zeros

Even though they describe different processes, the fact that gluon and graviton emission expressions were obtained from dimensional reduction of Yang-Mills and pure gravity amplitudes, points towards a similar nature of the radiation zeros in the planar regime.

In particular, the planar expansion of the factorized reduced pfaffian $E^{\phi\bar{\phi}g}(z,p,\varepsilon) = \text{Pf'}[\hat{\Psi}]\text{Pf}[X]$ inside the polarization factor reads the following way:

Scalar factor: $\mathbf{Pf}[X]\big|_{\mathbf{planar}}$. The indistinguishable case in Eq. (4.5.10) and the distinguishable case in Eq. (4.5.12) lead to the following expressions

$$\begin{aligned}
\text{Pf}[X]_{\text{indist}}(\sigma^{(1)},p)\big|_{\text{planar}} &\approx \frac{4\beta_1(1+\beta_2)}{\sigma_p}\left[\frac{1}{\beta_1 q_2^\perp} - \frac{1}{\eta(1+\beta_2)q_1^\perp} - \frac{1}{\beta_1 q_2^\perp - \eta(1+\beta_2)q_1^\perp}\right.\\
&\left.\quad + i\epsilon\left(\frac{\beta_1 q_2^\perp}{\left(\beta_1 q_2^\perp - \eta(1+\beta_2)q_1^\perp\right)^2} - \frac{1}{\beta_1 q_2^\perp}\right)\right] ,\\
\text{Pf}[X]_{\text{indist}}(\sigma^{(2)},p)\big|_{\text{planar}} &= \left(\text{Pf}[X]_{\text{indist}}(\sigma^{(1)},p)\big|_{\text{planar}}\right)^* , \qquad (4.6.18)\\
\text{Pf}[X]_{\text{dist}}(\sigma^{(1)},p)\big|_{\text{planar}} &\approx \frac{4(1+\beta_2)}{q_2^\perp \times \sigma_p}(1-i\epsilon) ,\\
\text{Pf}[X]_{\text{dist}}(\sigma^{(2)},p)\big|_{\text{planar}} &= \left(\text{Pf}[X]_{\text{dist}}(\sigma^{(1)},p)\big|_{\text{planar}}\right)^* . \qquad (4.6.19)
\end{aligned}$$

We can still visualize the different 'channels' for both of them and the limit turns out to be the same independently of the distinguishability of the particles i.e. $\text{Pf}[X]\big|_{\text{planar}} \sim \mathcal{O}(1) + i\mathcal{O}(\epsilon)$.

Scalar factor: $\mathbf{Pf'}[\hat{\Psi}]\big|_{\mathbf{planar}}$. This reduced pfaffian is always present inside the polarization factor and is never affected by the flavor configuration of the scalars. Its expansion, directly from Eq. (4.5.6), reads

$$\text{Pf'}[\hat{\Psi}](\sigma^{(1)},p,\varepsilon^-)\big|_{\text{planar}} = 0 , \qquad (4.6.20)$$

$$\begin{aligned}
\text{Pf'}[\hat{\Psi}](\sigma^{(2)},p,\varepsilon^-)\big|_{\text{planar}} &\approx \frac{\sqrt{2}\,\eta q_1^\perp q_2^\perp \beta_1(\beta_1-\beta_2)^2(1+\beta_2)}{\left(q_2^\perp - \eta q_1^\perp\right)\left(\beta_1 q_2^\perp - \beta_2\eta q_1^\perp\right)\left((1+\beta_1)q_2^\perp - \eta(1+\beta_2)q_1^\perp\right)\times\sigma_p}\\
&\quad\times\left[-i\epsilon + \epsilon^2\left(3 + \frac{\eta q_1^\perp}{q_2^\perp - \eta q_1^\perp} + \frac{\eta\beta_2 q_1^\perp}{\beta_1 q_2^\perp - \eta\beta_2 q_1^\perp}\right.\right.\\
&\quad\left.\left. + \frac{\eta(1+\beta_2)q_1^\perp}{(1+\beta_1)q_2^\perp - \eta(1+\beta_2)q_1^\perp}\right)\right] . \qquad (4.6.21)
\end{aligned}$$

Similarly to what happened for Yang-Mills MHV amplitudes, the second solution contribution reveals a single zero at leading order coming from the $\hat{Q}_1\hat{Q}_2^* - \hat{Q}_1^*\hat{Q}_2$ factor in the numerator i.e. $\text{Pf'}[\hat{\Psi}]\big|_{\text{planar}} \sim \mathcal{O}(\epsilon^2) + i\mathcal{O}(\epsilon)$. More specifically, comparing the two polarization factors

$$\left.\begin{array}{lll} \text{- Gluon emission case:} & \text{Pf'}[\hat{\Psi}]\text{Pf}[X](\sigma^{(2)}, p, \varepsilon^-)\Big|_{\text{planar}} \,, \\ \text{- Pure gluon case:} & E(\sigma^{(2)}, p, \text{MHV})\Big|_{\text{planar}} \,, \end{array}\right\} \sim \mathcal{O}(\epsilon^2) + i\mathcal{O}(\epsilon) \;. \tag{4.6.22}$$

Therefore, despite having a different analytic structure, gluon and graviton emission amplitudes share the same leading behavior in the planar regime as pure gluon and graviton amplitudes respectively

$$\mathcal{A}^{\phi\bar{\phi}g}_{\text{dist/indist}}\Big|_{\text{planar}} \sim \mathcal{O}(1) \;, \qquad \mathcal{M}^{\Phi\bar{\Phi}G}_{\text{dist/indist}}\Big|_{\text{planar}} \sim i\mathcal{O}(\epsilon) \;. \tag{4.6.23}$$

Trivially, graviton emission amplitudes vanish in this regime without any further kinematic condition. Moreover, in order to obtain the corresponding constraint defining gluon emission planar zeros, we have that the Parke-Taylor factor becomes again the relevant factor, since neither the jacobian nor the polarization factor give rise to any zero in the physical region. We present again the condition to emphasize the importance of the result

$$\begin{aligned} &c_{15432}\frac{\zeta_k - \zeta_{p'}}{\zeta_{p'}} + c_{14352}\frac{\zeta_{q'} - \zeta_k}{\zeta_k} - c_{13452}\frac{\zeta_{p'} - \zeta_k}{\zeta_k} \\ &\qquad + c_{13542}\frac{\zeta_{p'} - \zeta_{q'}}{\zeta_{q'}} - c_{15342}\frac{\zeta_k - \zeta_{q'}}{\zeta_{q'}} - c_{14532}\frac{\zeta_{q'} - \zeta_{p'}}{\zeta_{p'}} = 0 \;. \end{aligned} \tag{4.6.24}$$

Notice that we cannot make any direct comparison with the result in Eq. (2.3.9) concerning the scattering of distinguishable scalars due to the different color representation of the particles.

4.7 Closing remarks

We have presented a first complete analysis of the use of Sudakov variables in the context of the CHY calculation of scattering amplitudes. These amplitudes are represented as integrals with support on the solution to the SE. We saw that the final expression for pure scalar amplitudes has a simple formulation, given in terms of the position of the punctures. Likewise, even with less naive theories involving non-trivial interactions such us Yang-Mills or Einstein-Hilbert gravity, the Sudakov representation has been shown to adopt compact formulae when studying the analytical structure of the different building blocks constituting the amplitudes; especially in the case of the pfaffians, whose complexity rapidly grows and becomes inaccesible as the multiplicity increases.

We have been able to access the planar behavior of all the amplitudes, unveiling the details of the same planar radiation zeros discussed throughout Chapters 1 and 2. First of all, we identified planar graviton configurations to vanish due to a double-zero contribution in the polarization factor $E(z, p, \varepsilon)^2$. The same applies for the emission of a single graviton from two scalars due to the same structure of the pfaffians, suggesting that this feature is present in any process of a spin-2 theory. Similarly, we found gluon planar zeros to be characterized just by constraining the Parke-Taylor factor $C(\rho, z)$, which only depends on the partial ordering of the gluons in the full amplitude. This result is remarkable since it leads to the same condition for the gluon emission, regardless of the nature of the particles involved in the process. Both gauge and gravitational amplitudes are built as a single term coming from one of the rational solutions to the SE. Last, we checked that biadjoint scalar amplitudes, receiving contributions from all the SE solutions, give rise to a different planar zero condition with a non-projective structure.

In a future work we will generalize the Sudakov representation for a n-point amplitude and show the different multi-particle factorization limits which are naturally parametrized in this approach. The connection among gravitational and Yang-Mills amplitudes in this approach from the point of view of Regge kinematics [46, 48, 51, 119] is also of interest, together with the corresponding soft theorems [21, 120–124]. Besides, it would be interesting to interpret the role of the gluing operator recently investigated in [125] in terms of Sudakov variables. Certainly, the relevance of this operator for the calculation of higher-loop amplitudes is still to be investigated and exploring kinematical limits, such as multi-Regge kinematics where the Sudakov representation is most useful, could be a possible route to understand its meaning.

CONCLUSIONS / CONCLUSIONES

The study of the mathematical structure of scattering amplitudes has made profound progress in the recent years, pointing towards more efficient computational methods and allowing for the discovery of intriguing symmetries and dualities not manifest in the standard quantum field theory (QFT) approach. Some examples are the so-called *color-kinematics duality*, used throughout this thesis as a procedure to compute gravitational amplitudes from their gauge analogues; and the *Cachazo-He-Yuan (CHY) formalism*, as a novel integral representation to write scattering amplitudes that circumvents all gauge redundancies naturally present in the traditional Feynman diagram decomposition. The latter relies upon a rational map between the space of null D-dimensional momentum vectors and the moduli space of punctured Riemann spheres, given the name of *scattering equations (SE)*. Being a rather hard task, obtaining the analytic solutions to the SE is needed in order to write the complete expression of scattering amplitudes. In this work we have shown for the first time the advantages of using the Sudakov parametrization of particle momenta to simplify their computation. In particular, we have studied in detail the formulas of Fairlie's solution in the four- and five-point cases. The corresponding punctures, written in terms of Sudakov variables, turn out to adopt compact expressions that can be interpreted as the stereographic projection of the momentum vectors onto the unit sphere. Moreover, partially fixing the $SL(2, \mathbb{C})$ symmetry involved in the problem, we have found these punctures to live on circles parametrized by a single Sudakov variable in the four-point case, and by four Sudakov variables in the five-point case. The six-point case is a bit more complicated, but the use of Sudakov variables has allowed us to obtain explicit expressions for the modulus of the whole set of exact solutions to the SE, which have never been shown in the literature until now. All these results have been discussed in Chapter 3. Additionally, in Chapter 4, we have studied the expressions of scattering amplitudes themselves inside the biadjoint scalar theory, Yang-Mills theory and Einstein-Hilbert gravity, finding a similar result: explicit formulas turn out to notably simplify when expressed in terms of Sudakov variables, suggesting that the parametrization is a natural candidate for an efficient description of scattering amplitudes inside the CHY formalism.

All the framework being established, we have presented a detailed description of *planar radiation zeros* as a novel mathematical structure giving rise to new insights on the internal behavior of a theory. Concretely, we have studied their appearance in biadjoint scalar theory, Yang-Mills theory, Einstein-Hilbert gravity and in two extensions of the latter including scalar particles. The concept "radiation zero" makes reference to all the configurations in phase space for which the full scattering amplitude of a given process vanishes. In our case, we have studied "planar zeros", meaning that our characterization applies to those processes where all particle momenta lie in the same spatial plane. Although being a rather naive concept, the obtained results are far from incidental. In Chapters 1 and 2, we have found that the conditions of emergence of the n-gluon planar zeros in the *maximally helicity violating (MHV)* sector live inside the projective space spanned by the stereographic coordinates labelling the direction of flight of the outgoing momenta. Moreover, planar zeros arising from the gluon emission process enjoy the same feature. The existence of such a projective characterization in gauge theories implies that planar zeros are always realized inside the soft limit of one of the emitted particles, which might be of relevance for the infrared structure of the theory. Likewise, it would be interesting to explore whether planar zeros are of any relevance for the asymptotic symmetries of the theory. On a different side, we have found that gravitational amplitudes always vanish inside the planar limit for non-helicity conserving configurations without imposing any further kinematic conditions. The reason for such a peculiar behavior is that, in a sense, planar gravitational zeros are just mimicking the properties of three-dimensional gravities, whose amplitudes identically vanish by symmetry arguments over their helicity structure. Planar zeros have also been studied in detail for pure scalar amplitudes and for the string α'-corrections of the formers, concluding that any of the mentioned properties translate into these cases. Finally, coming back to the results of Chapter 4, it is worth mentioning that the CHY formalism under the Sudakov parametrization has allowed us to study the nature of the planar zeros discussed above, unveiling their emergence right at the level of the integrand. Some implications are straightforward, such as the fact that all gauge processes share the same planar zero condition regardless of their matter content, provided that all particles transform under the same representation of the gauge group; or why all gravitational amplitudes exactly vanish independently of the nature of the particles involved.

* * *

El estudio de la estructura matemática en amplitudes de dispersión ha tenido grandes avances en los últimos años, dando lugar a métodos computacionales más eficientes y permitiendo el descubrimiento de simetrías y dualidades que no son evidentes desde el punto de vista de la Teoría Cuántica de Campos. Algunos ejemplos son la *dualidad color-cinemática*, utilizada a lo largo de esta memoria como un procedimiento para calcular amplitudes gravitatorias a partir de sus análogas en teorías 'gauge'; y el *formalismo de Cachazo-He-Yuan (CHY)*, como una representación novedosa libre de redundancias 'gauge' —normalmente presentes en representaciones más tradicionales como la descomposición en diagramas de Feynman— con la que escribir las amplitudes de dispersión. Este formalismo está fuertemente basado en un mapa entre el espacio de momentos nulos D-dimensionales y el espacio modular de esferas de Riemann con punturas, conocido como *ecuaciones de dispersión (SE)*. A pesar de ser una tarea ardua, la obtención de todas las soluciones analíticas a estas ecuaciones es necesaria para poder escribir las expresiones completas de las amplitudes de dispersión. En este trabajo, hemos mostrado por primera vez las ventajas de utilizar la parametrización de Sudakov en el espacio de momentos para simplificar este cálculo. En particular, hemos estudiado en detalle las fórmulas de la solución de

Fairlie a cuatro y cinco puntos. Las correspondientes punturas, escritas en función de las variables de Sudakov, resultan ser expresiones muy compactas que pueden interpretarse como la proyección estereográfica de los momentos sobre la esfera unidad. Además, fijando parcialmente la simetría $SL(2,\mathbb{C})$ del problema, hemos encontrado que estas punturas viven sobre circunferencias en la esfera parametrizadas por una sola variable de Sudakov en el caso a cuatro puntos, y por cuatro variables de Sudakov en el caso a cinco puntos. El caso a seis puntos es algo más complicado, pero el uso de las variables de Sudakov nos ha permitido obtener expresiones explícitas de los módulos de las soluciones exactas a las ecuaciones de dispersión, las cuáles nunca han sido recogidas en la literatura. Todos estos resultados se discuten en el capítulo 3. Además, en el capítulo 4, hemos estudiado las expresiones de las amplitudes de dispersión dentro de la teoría de escalares biadjuntos, teoría de Yang-Mills y gravedad de Einstein-Hilbert, encontrando un resultado similar: las fórmulas explícitas se simplifican notablemente al escribirlas en función de las variables de Sudakov. Esto sugiere que esta parametrización es una candidata natural para una descripción eficiente de las amplitudes de dispersión dentro del formalismo de CHY.

Una vez establecido todo el marco técnico, hemos presentado una descripción detallada de los *ceros de radiación planares*, como una estructura matemática original que aporta nuevos puntos de vista sobre el comportamiento interno de una teoría. Concretamente, hemos estudiado sus propiedades en la teoría de escalares biadjuntos, la teoría de Yang-Mills, la gravedad de Einstein-Hilbert y en dos extensiones de las anteriores que incluyen la presencia de partículas escalares. El concepto de "cero de radiación" hace referencia a las configuraciones del espacio de fases para las que la amplitud de dispersión completa de un proceso dado se anula. En nuestro caso, hemos estudiado "ceros planares", lo que quiere decir que hemos realizado nuestra caracterización en procesos para los que todos los momentos de las partículas involucradas se encuentran confinados dentro de un mismo plano espacial. A pesar de ser un concepto muy sencillo, los resultados obtenidos están lejos de ser casuales. En los capítulos 1 y 2, hemos encontrado que las condiciones de emergencia de ceros planares para n gluones en el sector que viola helicidad máximalmente, se encuentran definidas dentro de un espacio proyectivo generado por las coordenadas estereográficas correspondientes a la dirección espacial de los momentos de las partículas finales. Además, los ceros planares surgidos de la emisión de un gluon también disfrutan de esta característica. La existencia de esta caracterización en un espacio proyectivo para teorías 'gauge' implica que los ceros planares tienen lugar siempre en el límite en que una de las partículas emitidas tiene poca energía, lo que puede tener importancia para la estructura en el infrarrojo de la teoría. De la misma forma, sería interesante explorar si estos ceros planares tienen alguna relevancia para el estudio de las simetrías asintóticas de la teoría. Desde otro punto de vista, hemos encontrado que las amplitudes gravitatorias siempre se anulan en el límite planar para configuraciones que no conservan helicidad, sin la necesidad de imponer ninguna condición cinemática adicional. La razón detrás de este comportamiento es que, en cierto sentido, los ceros planares gravitatorios simplemente imitan las propiedades de gravedades tres-dimensionales, cuyas amplitudes se anulan idénticamente en base a argumentos de simetría sobre su estructura de helicidad. Los ceros planares también se han estudidado en detalle para amplitudes puramente escalares y para correcciones en teoría de cuerdas de las ya mencionadas, concluyendo que ninguno de los resultados ya mencionados aparece en estos casos. Finalmente, volviendo sobre los resultados del capítulo 4, cabe mencionar que el formalismo de CHY y el uso de la parametrización de Sudakov han permitido el estudio de la naturaleza de los ceros planares discutidos más arriba, descubriendo su origen a nivel de integrando. Algunas inferencias son directas, como por ejemplo el hecho de que todos los procesos 'gauge' compartan los mismos ceros planares independientemente del contenido en materia de la teoría, siempre que todas las partículas transformen bajo la misma representación del grupo 'gauge'; o por qué las amplitudes gravitatorias se anulan de forma exacta independientemente de la naturaleza de las partículas involucradas.

Appendix A

Scattering Amplitudes Review

A.1 Spinor-helicity formalism

Throughout most of the traditional Quantum Field Theory courses, four-momentum vectors p^μ are always treated as the standard variables for the kinematic description of any process. Indeed, using momentum vectors is a rather natural way to visualize and understand a particle collision. However, it turns out not to be the most convenient way of writing scattering amplitudes in general, specially whenever any of the particles carry non-zero spin. In such a case polarization vectors ε^μ must be included in the computations in order to build Lorentz invariant quantities, with the inevitable introduction of many gauge redundancies that entail quite lengthy and unnecessary expressions.

The 'spinor-helicity formalism' is used as a simplifying operational tool. Basically, it considers different basic building blocks for the construction of the amplitudes which, although less intuitive, are in general more efficient and practical for computations. The formalism just plays around with a smaller representation of the Lorentz group, providing with a really nice and useful framework to describe scattering amplitudes of massless particles for arbitrary helicities. Recent reviews can be found in [4–7, 126]. In particular, it exploits the local isomorphism $SO(3,1) \cong SL(2,\mathbb{C})$ to write every element in spinor representation. Four-momentum vectors for example translate into the following (2×2)-matrix with spinor indices

$$p^\mu \to p^{\dot\alpha\alpha} = \bar\sigma_\mu^{\dot\alpha\alpha} p^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} , \tag{A.1.1}$$

where $\bar\sigma_\mu^{\dot\alpha\alpha} = \left(\mathbb{I}^{\dot\alpha\alpha}, \sigma_x^{\dot\alpha\alpha}, \sigma_y^{\dot\alpha\alpha}, \sigma_z^{\dot\alpha\alpha}\right)$ are the Pauli matrices. Notice that undotted α and dotted $\dot\alpha$ spinor indices transform according to different $SL(2,\mathbb{C})$ representations —i.e. fundamental and anti-fundamental respectively—. That is why they are also called 'chiral indices'. The onshellness of the particle is then written as a condition over the determinant

$$p^2 = m^2 \quad \Rightarrow \quad \det(p^{\dot\alpha\alpha}) = m^2 , \tag{A.1.2}$$

which vanishes in the case of massless particles ($\det(p^{\dot\alpha\alpha}) = 0$). This is a crucial point since, according to the rank of the matrix, it allows to factorize the momentum matrix into a product of two 2-dimensional vectors

$$\det\left(p^{\dot{\alpha}\alpha}\right) = 0 \quad \Rightarrow \quad p^{\dot{\alpha}\alpha} = \tilde{\lambda}^{\dot{\alpha}}\lambda^{\alpha} \; . \tag{A.1.3}$$

These two Weyl spinors, up to a phase factor ($\{\lambda^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}\} \mapsto \{e^{i\varphi}\lambda^{\alpha}, e^{-i\varphi}\tilde{\lambda}^{\dot{\alpha}}\}$) that leaves the momentum matrix in eq.(A.1.1) invariant, can be defined as

$$\lambda^{\alpha} = \frac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^0+p^3 \\ p^1+ip^2 \end{pmatrix} \; , \qquad \tilde{\lambda}^{\dot{\alpha}} = \frac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^0+p^3 \\ p^1-ip^2 \end{pmatrix} \; , \tag{A.1.4}$$

with the clear relation between them of conjugacy $(\lambda^{\alpha})^* = \tilde{\lambda}^{\dot{\alpha}}$ for real momenta[1]. Sometimes a more uniform prescription is preferred where all four-momenta are taken to be incoming. As a consequence, all outgoing particles are prescribed to have negative energy, i.e. $p^0 \pm p^3|_{\text{out}} < 0$, introducing some ambiguity into the previous definition. The correct evaluation of the square roots in such a case must be

$$\sqrt{p^0+p^3} \quad \mapsto \quad \begin{cases} \sqrt{p^0+p^3} & \text{for} \quad p^0 > 0 \; , \\ i\sqrt{|p^0+p^3|} & \text{for} \quad p^0 < 0 \; , \end{cases} \tag{A.1.5}$$

meaning that the relation between spinors is then $(\lambda^{\alpha})^* = \text{sgn}(p^0)\,\tilde{\lambda}^{\dot{\alpha}}$.

These new objects λ^{α} and $\tilde{\lambda}^{\dot{\alpha}}$, refered as 'helicity spinors' from now on, become in this way the basic ingredients of the formalism for the description of massless[2] particles. Once they are presented, it is time therefore to start manipulating them. The first step is to define a way of raising and lowering the spinor indices. This can be done throught the antisymmetric Levi-Civita tensor

$$\lambda_{\alpha} := \epsilon_{\alpha\beta}\lambda^{\beta} \; , \qquad \tilde{\lambda}_{\dot{\alpha}} := \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}} \; . \tag{A.1.6}$$

Hence, the most direct Lorentz-invariant objects that can be constructed out of them are just the spinor contractions

$$\langle ij\rangle := \begin{pmatrix} \lambda_i^{\alpha} & 0 \end{pmatrix}\begin{pmatrix} \lambda_{j,\alpha} \\ 0 \end{pmatrix} = \epsilon_{\alpha\beta}\lambda_i^{\alpha}\lambda_j^{\beta} \; , \qquad [ij] := \begin{pmatrix} 0 & \tilde{\lambda}_{i,\dot{\alpha}} \end{pmatrix}\begin{pmatrix} 0 \\ \tilde{\lambda}_j^{\dot{\alpha}} \end{pmatrix} = -\epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_j^{\dot{\beta}} \; , \tag{A.1.7}$$

where a new angle ($\langle\cdot|,|\cdot\rangle$) and square ($[\cdot|,|\cdot]$) bracket notation has been introduced for simplicity. They are both related by complex conjugation as $[ij] = -\text{sgn}(p_i^0 p_j^0)\langle ij\rangle^*$. Notice the different sign convention according to the representation of the spinor.

Some of the properties for these contractions can be read directly from the definition

[1] It is common in the field of 'Modern Methods for Scattering Amplitudes' [4–7, 9] to enhance the description of four-momentum vectors into a complex vector space. In that case λ^{α} and $\tilde{\lambda}^{\dot{\alpha}}$ are simply independent spinors.

[2] There also exist some generalizations of the formalism for the case of massive particles. A nice introduction can be found in [127].

by means of the antisymmetric tensor

$$\langle ij\rangle = -\langle ji\rangle\ , \qquad [ij] = -[ji]\ , \qquad \langle ii\rangle = [ii] = 0\ . \tag{A.1.8}$$

Less straightforward but really useful in the translation from the $SO(3,1)$ to the $SL(2,\mathbb{C})$ representation of the Lorentz group is the expression for the Mandelstam invariants or general four-momentum vectors

$$s_{ij} = 2p_i\cdot p_j = \langle ij\rangle[ji]\ , \qquad 2p_i^\mu = \lambda_i^\alpha \sigma^\mu_{\alpha\dot\alpha}\tilde\lambda_i^{\dot\alpha} \equiv \langle i|\gamma^\mu|i]\ , \tag{A.1.9}$$

with the Dirac matrices γ^μ inside the bracket notation considered in the chiral representation in order to be consistent with the spinor indices contraction. This last equation tells us, together with the conjugacy relation between spinors, that the modulus of any of the brackets is $|\langle ij\rangle| = |[ij]| = \sqrt{|s_{ij}|}$. There are also two other powerful identities coming from total momentum conservation

$$\sum_{k=1}^{n}\lambda_k^\alpha\tilde\lambda_k^{\dot\alpha} = 0 \qquad \Rightarrow \qquad \sum_{k=1}^{n}\langle ik\rangle[kj] = 0\ , \tag{A.1.10}$$

and from the Schouten identity for the 2-dimensional antisymmetric tensor

$$\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}+\epsilon_{\beta\gamma}\epsilon_{\alpha\delta}+\epsilon_{\gamma\alpha}\epsilon_{\beta\delta} = 0 \qquad \Rightarrow \qquad \begin{cases} \langle ij\rangle\langle kl\rangle + \langle jk\rangle\langle il\rangle + \langle ki\rangle\langle jl\rangle &= 0\ , \\ [ij][kl] + [jk][il] + [ki][jl] &= 0\ . \end{cases} \tag{A.1.11}$$

Up to now, only momentum vectors and their Lorentz-invariant contractions have been translated into the spinor-helicity framework. However, polarization vectors adopt as well a really nice form in this representation

$$\varepsilon_+^{\dot\alpha\alpha}(p) = -\sqrt{2}\,\frac{\tilde\lambda_p^{\dot\alpha}\lambda_q^\alpha}{\langle pq\rangle}\ , \qquad \varepsilon_-^{\alpha\dot\alpha}(p) = \sqrt{2}\,\frac{\lambda_p^\alpha\tilde\lambda_q^{\dot\alpha}}{[pq]}\ . \tag{A.1.12}$$

Indeed, all the construction of the formalism has its origin in a series of papers from the 1980s [128–132], where they first realized that polarization vectors for massless vector particles with definite helicity admit the following decomposition in terms of Weyl spinors

$$\varepsilon_+^\mu(p) = -\frac{\lambda_q^\alpha\sigma^\mu_{\alpha\dot\alpha}\tilde\lambda_p^{\dot\alpha}}{\sqrt{2}\,\langle pq\rangle} \equiv -\frac{[p|\gamma^\mu|q\rangle}{\sqrt{2}\,\langle pq\rangle}\ , \qquad \varepsilon_-^\mu(p) = \frac{\lambda_p^\alpha\sigma^\mu_{\alpha\dot\alpha}\tilde\lambda_q^{\dot\alpha}}{\sqrt{2}\,[pq]} \equiv \frac{\langle p|\gamma^\mu|q]}{\sqrt{2}\,[pq]}\ . \tag{A.1.13}$$

In both expressions Eq. (A.1.12) and Eq. (A.1.13), the arbitrariness of momentum vector q^μ has to do with the gauge redundancy encoded in the polarization of the particle. In this representation, the usual properties for the polarization vectors become manifest

$$(\varepsilon_+^{\alpha\dot\alpha})^* = \varepsilon_-^{\alpha\dot\alpha}\ ,$$

$$
\begin{aligned}
p\cdot\varepsilon_+(p) &= \frac{1}{2}\lambda_\alpha\tilde{\lambda}_{\dot\alpha}\varepsilon_+^{\alpha\dot\alpha} \sim [pp] = 0\ ,\\
p\cdot\varepsilon_-(p) &= \frac{1}{2}\lambda_\alpha\tilde{\lambda}_{\dot\alpha}\varepsilon_-^{\alpha\dot\alpha} \sim \langle pp\rangle = 0\ ,\\
\varepsilon_+(p)^{\dot\alpha\alpha}\varepsilon_-(p)_{\alpha\dot\alpha} &= -\frac{(\tilde\lambda_p)^{\dot\alpha}(\lambda_q)^\alpha(\lambda_p)_\alpha(\tilde\lambda_q)_{\dot\alpha}}{\langle pq\rangle[pq]} = -1\ ,\\
\varepsilon_+(p)^{\dot\alpha\alpha}\varepsilon_+(p)_{\alpha\dot\alpha} &= \frac{(\tilde\lambda_p)^{\dot\alpha}(\lambda_q)^\alpha(\tilde\lambda_p)_{\dot\alpha}(\lambda_q)_\alpha}{\langle pq\rangle^2} = 0\ .
\end{aligned}
\tag{A.1.14}
$$

Finally, it is important to remark that 'helicity spinors' —honoring their name— directly carry information as well about the helicities of the particles. This can be seen from the phase invariance refered above in their definition in Eq. (A.1.4). The $U(1)$ little group is generated by the helicity operator

$$
h = \frac{1}{2}\sum_{i=1}^{n}\left[-\lambda_i^\alpha\frac{\partial}{\partial\lambda_i^\alpha} + \tilde\lambda_i^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\right]\ . \tag{A.1.15}
$$

Acting on every spinor, it turns out that they individually represent each of the helicities a particle can have, receiving thus the alternative name of 'helicity basis'

$$
h\lambda^\alpha = -\frac{1}{2}\lambda^\alpha\ , \qquad h\tilde\lambda^{\dot\alpha} = +\frac{1}{2}\tilde\lambda^{\dot\alpha}\ . \tag{A.1.16}
$$

As expected, one can recover from here the correct helicity for the polarization vectors

$$
h\ \varepsilon_\pm^{\alpha\dot\alpha}(p) = (\pm 1)\ \varepsilon_\pm^{\alpha\dot\alpha}(p)\ . \tag{A.1.17}
$$

Summing up, it has been described how the 'spinor-helicity formalism' turns out to be a great framework for the description of the degrees of freedom inside on-shell massless amplitudes. A glimpse of this is caught in the Preface, where usefulness of Parke-Taylor formulae for the description of multi-parton amplitudes [9, 13] is presented. However, apart from allowing more compact and tractable expressions than the traditional four-vectors, the formalism takes even more relevance when realizing that the Lorentz little group takes a particularly simple form in spinor representation, making much easier to impose the corresponding symmetry constraints for the construction of the three-point amplitude [133].

Appendix B

Complementary material

B.1 The numerator $P_{10}(\zeta_3, \zeta_4, \zeta_5)$ in equation (2.2.7)

Here we give the explicit expression of the numerator $P_{10}(\zeta_3, \zeta_4, \zeta_5)$ in Eq. (2.2.7), for generic color factors

$$
\begin{aligned}
P_{10}(\zeta_3, \zeta_4, \zeta_5) = {} & c_7\bar{c}_7\zeta_3^6\zeta_4^3\zeta_5 + (c_{15}\bar{c}_{15} - c_7\bar{c}_7 - c_8\bar{c}_8)\zeta_3^6\zeta_4^2\zeta_5^2 \\
& + c_8\bar{c}_8\zeta_3^6\zeta_4\zeta_5^3 - c_7\bar{c}_7\zeta_3^5\zeta_4^4\zeta_5 + (2c_8\bar{c}_8 - c_7\bar{c}_7 - 2c_{15}\bar{c}_{15})\zeta_3^5\zeta_4^3\zeta_5^2 \\
& + c_7\bar{c}_7\zeta_3^5\zeta_4^3 + (2c_7\bar{c}_7 - c_8\bar{c}_8 - 2c_{15}\bar{c}_{15})\zeta_3^5\zeta_4^2\zeta_5^3 - c_7\bar{c}_7\zeta_3^5\zeta_4^2\zeta_5 \\
& - c_8\bar{c}_8\zeta_3^5\zeta_4\zeta_5^4 - c_8\bar{c}_8\zeta_3^5\zeta_4\zeta_5^2 + c_8\bar{c}_8\zeta_3^5\zeta_5^3 - c_6\bar{c}_6\zeta_3^4\zeta_4^5\zeta_5 \\
& + (c_{10}\bar{c}_{10} - c_{13}\bar{c}_{13} + c_{15}\bar{c}_{15} - c_3\bar{c}_3 + c_5\bar{c}_5 + 2c_6\bar{c}_6 + 2c_7\bar{c}_7 - c_8\bar{c}_8)\zeta_3^4\zeta_4^4\zeta_5^2 \\
& + (c_2\bar{c}_2 + c_3\bar{c}_3 - c_1\bar{c}_1 - 2c_{10}\bar{c}_{10} - c_{11}\bar{c}_{11} + c_{12}\bar{c}_{12} + c_{13}\bar{c}_{13} \\
& + 4c_{15}\bar{c}_{15} - 2c_5\bar{c}_5 - c_6\bar{c}_6 - c_7\bar{c}_7 - c_8\bar{c}_8)\zeta_3^4\zeta_4^3\zeta_5^3 \\
& + (c_{14}\bar{c}_{14} - c_6\bar{c}_6 - c_7\bar{c}_7)\zeta_3^4\zeta_4^4 + (2c_6\bar{c}_6 - c_7\bar{c}_7 - 2c_{14}\bar{c}_{14})\zeta_3^4\zeta_4^3\zeta_5 \\
& + (c_{10}\bar{c}_{10} + 2c_{11}\bar{c}_{11} - c_{12}\bar{c}_{12} + c_{15}\bar{c}_{15} - c_2\bar{c}_2 + c_5\bar{c}_5 - c_7\bar{c}_7 + 2c_8\bar{c}_8)\zeta_3^4\zeta_4^2\zeta_5^4 \\
& + (c_4\bar{c}_4 - c_1\bar{c}_1 - c_{11}\bar{c}_{11} + c_{14}\bar{c}_{14} - c_6\bar{c}_6 + 2c_7\bar{c}_7 + 2c_8\bar{c}_8 + c_9\bar{c}_9)\zeta_3^4\zeta_4^2\zeta_5^2 \\
& - c_{11}\bar{c}_{11}\zeta_3^4\zeta_4\zeta_5^5 + (2c_{11}\bar{c}_{11} - c_8\bar{c}_8 - 2c_9\bar{c}_9)\zeta_3^4\zeta_4\zeta_5^3 + (c_9\bar{c}_9 - c_8\bar{c}_8 - c_{11}\bar{c}_{11})\zeta_3^4\zeta_5^4 \\
& + c_6\bar{c}_6\zeta_3^3\zeta_4^6\zeta_5 + (2c_{13}\bar{c}_{13} - c_6\bar{c}_6 - 2c_{10}\bar{c}_{10})\zeta_3^3\zeta_4^5\zeta_5^2 + c_6\bar{c}_6\zeta_3^3\zeta_4^5 \\
& + (c_1\bar{c}_1 + 4c_{10}\bar{c}_{10} + c_{11}\bar{c}_{11} - c_{12}\bar{c}_{12} - c_{13}\bar{c}_{13} - 2c_{15}\bar{c}_{15} - c_2\bar{c}_2 + c_3\bar{c}_3 \\
& - 2c_5\bar{c}_5 - c_6\bar{c}_6 - c_7\bar{c}_7 + c_8\bar{c}_8)\zeta_3^3\zeta_4^4\zeta_5^3 + (2c_7\bar{c}_7 - c_6\bar{c}_6 - 2c_{14}\bar{c}_{14})\zeta_3^3\zeta_4^4\zeta_5 \\
& + (c_1\bar{c}_1 - 2c_{10}\bar{c}_{10} - c_{11}\bar{c}_{11} + c_{12}\bar{c}_{12} - c_{13}\bar{c}_{13} - 2c_{15}\bar{c}_{15} - c_2\bar{c}_2 - c_3\bar{c}_3 \\
& + 4c_5\bar{c}_5 + c_6\bar{c}_6 + c_7\bar{c}_7 - c_8\bar{c}_8)\zeta_3^3\zeta_4^3\zeta_5^4 + (c_1\bar{c}_1 + c_{11}\bar{c}_{11} + c_{12}\bar{c}_{12} - c_{13}\bar{c}_{13} \\
& + 4c_{14}\bar{c}_{14} + c_2\bar{c}_2 - c_3\bar{c}_3 - 2c_4\bar{c}_4 - c_6\bar{c}_6 - c_7\bar{c}_7 - c_8\bar{c}_8 - 2c_9\bar{c}_9)\zeta_3^3\zeta_4^3\zeta_5^2 \\
& + (2c_2\bar{c}_2 - 2c_5\bar{c}_5 - c_{11}\bar{c}_{11})\zeta_3^3\zeta_4^2\zeta_5^5 + (c_1\bar{c}_1 - c_{11}\bar{c}_{11} - c_{12}\bar{c}_{12} + c_{13}\bar{c}_{13}
\end{aligned}
$$

$$
\begin{aligned}
&- 2c_{14}\bar{c}_{14} - c_2\bar{c}_2 + c_3\bar{c}_3 - 2c_4\bar{c}_4 + c_6\bar{c}_6 - c_7\bar{c}_7 - c_8\bar{c}_8 + 4c_9\bar{c}_9)\zeta_3^3\zeta_4^2\zeta_5^3 \\
&+ c_{11}\bar{c}_{11}\zeta_3^3\zeta_4\zeta_5^6 + (2c_8\bar{c}_8 - 2c_9\bar{c}_9 - c_{11}\bar{c}_{11})\zeta_3^3\zeta_4\zeta_5^4 + c_{11}\bar{c}_{11}\zeta_3^3\zeta_5^5 \\
&+ (c_{10}\bar{c}_{10} - c_6\bar{c}_6 - c_{13}\bar{c}_{13})\zeta_3^2\zeta_4^6\zeta_5^2 + (2c_6\bar{c}_6 - 2c_{10}\bar{c}_{10} - c_{13}\bar{c}_{13})\zeta_3^2\zeta_4^5\zeta_5^3 \\
&- c_6\bar{c}_6\zeta_3^2\zeta_4^5\zeta_5 + (-c_1\bar{c}_1 + c_{10}\bar{c}_{10} - c_{11}\bar{c}_{11} + 2c_{13}\bar{c}_{13} + c_{15}\bar{c}_{15} \\
&+ 2c_2\bar{c}_2 + c_5\bar{c}_5 - c_6\bar{c}_6)\zeta_3^2\zeta_4^4\zeta_5^4 + (-c_{12}\bar{c}_{12} + 2c_{13}\bar{c}_{13} + c_{14}\bar{c}_{14} - c_2\bar{c}_2 + c_4\bar{c}_4 \\
&+ 2c_6\bar{c}_6 - c_7\bar{c}_7 + c_9\bar{c}_9)\zeta_3^2\zeta_4^4\zeta_5^2 + (-c_2\bar{c}_2 - 2c_5\bar{c}_5 + 2c_{11}\bar{c}_{11})\zeta_3^2\zeta_4^3\zeta_5^5 \\
&+ (-c_1\bar{c}_1 - c_{11}\bar{c}_{11} + c_{12}\bar{c}_{12} - c_{13}\bar{c}_{13} - 2c_{14}\bar{c}_{14} - c_2\bar{c}_2 + c_3\bar{c}_3 \\
&+ 4c_4\bar{c}_4 - c_6\bar{c}_6 + c_7\bar{c}_7 + c_8\bar{c}_8 - 2c_9\bar{c}_9)\zeta_3^2\zeta_4^3\zeta_5^3 + (-c_2\bar{c}_2 + c_5\bar{c}_5 - c_{11}\bar{c}_{11})\zeta_3^2\zeta_4^2\zeta_5^6 \\
&+ (2c_{11}\bar{c}_{11} - c_{13}\bar{c}_{13} + c_{14}\bar{c}_{14} + 2c_2\bar{c}_2 - c_3\bar{c}_3 + c_4\bar{c}_4 - c_8\bar{c}_8 + c_9\bar{c}_9)\zeta_3^2\zeta_4^2\zeta_5^4 \\
&- c_{11}\bar{c}_{11}\zeta_3^2\zeta_4\zeta_5^5 + c_{13}\bar{c}_{13}\zeta_3\zeta_4^6\zeta_5^3 - c_{13}\bar{c}_{13}\zeta_3\zeta_4^5\zeta_5^4 - c_{13}\bar{c}_{13}\zeta_3\zeta_4^5\zeta_5^2 - c_2\bar{c}_2\zeta_3\zeta_4^4\zeta_5^5 \\
&+ (2c_2\bar{c}_2 - 2c_4\bar{c}_4 - c_{13}\bar{c}_{13})\zeta_3\zeta_4^4\zeta_5^3 + c_2\bar{c}_2\zeta_3\zeta_4^3\zeta_5^6 + (-c_2\bar{c}_2 - 2c_4\bar{c}_4 + 2c_{13}\bar{c}_{13})\zeta_3\zeta_4^3\zeta_5^4 \\
&- c_2\bar{c}_2\zeta_3\zeta_4^2\zeta_5^5 + c_{13}\bar{c}_{13}\zeta_4^5\zeta_5^3 + (-c_2\bar{c}_2 + c_4\bar{c}_4 - c_{13}\bar{c}_{13})\zeta_4^4\zeta_5^4 + c_2\bar{c}_2\zeta_4^3\zeta_5^5 .
\end{aligned}
\tag{B.1.1}
$$

Due to Bose symmetry, the polynomial is invariant under permutations of its three variables ζ_3, ζ_4, and ζ_5, provided this is supplemented with the corresponding permutation of S_3 acting on the color factors, as explained in [1].

B.2 The coefficients $A_5^{(2)}$ and $A_5^{(3)}$ of the α' expansion (2.4.12)

The coefficient $A_5^{(2)}$ of the α'^2 correction to the five-gluon amplitude is a degree 10 polynomial in the stereographic coordinates, containing monomials of degree 8, 6, 4, and 2 as well

$$
\begin{aligned}
A_5^{(2)}(\zeta_3,\zeta_4,\zeta_5) = &-(c_6+c_7)\zeta_3\zeta_4 - (c_8+c_{11})\zeta_3\zeta_4 - (c_2+c_{13})\zeta_4\zeta_5 \\
&- 2(c_6+c_7+c_8+c_{11})\zeta_3^2\zeta_4\zeta_5 - 2(c_2+c_6+c_7+c_{13})\zeta_3\zeta_4^2\zeta_5 \\
&- 2(c_2+c_8+c_{11}+c_{13})\zeta_3\zeta_4\zeta_5^2 - 4(c_2+c_6+c_7+c_8+c_{11}+c_{13})\zeta_3^2\zeta_4^2\zeta_5^2 \\
&+ (c_2-c_7-c_{11}-c_{13})\zeta_3^3\zeta_4^2\zeta_5 + (-c_7+c_8-c_{11}-c_{13})\zeta_3\zeta_4^3\zeta_5^2 \\
&+ (-c_2-c_6-c_8+c_{11})\zeta_3^2\zeta_4^3\zeta_5 + (-c_2-c_6-c_8+c_{13})\zeta_3^3\zeta_4\zeta_5^2 \\
&+ (c_6-c_7-c_{11}-c_{13})\zeta_3^2\zeta_4\zeta_5^3 + (-c_2-c_6+c_7-c_8)\zeta_3\zeta_4^2\zeta_5^3 \\
&- 2(c_6+c_7+c_8+c_{13})\zeta_3^3\zeta_4^3\zeta_5^2 - 2(c_2+c_7+c_8+c_{11})\zeta_3^3\zeta_4^2\zeta_5^3 \\
&- 2(c_2+c_6+c_{11}+c_{13})\zeta_3^2\zeta_4^3\zeta_5^3 - (c_7+c_8)\zeta_3^4\zeta_4^3\zeta_5^3 - (c_6+c_{13})\zeta_3^3\zeta_4^4\zeta_5^3 \\
&- (c_2+c_{11})\zeta_3^3\zeta_4^3\zeta_5^4 .
\end{aligned}
\tag{B.2.1}
$$

The coefficient $A_5^{(3)}$ contains monomials of degree 15, 13, 11, 9, 7, 5, and 3

$$
\begin{aligned}
A_5^{(3)}(\zeta_3,\zeta_4,\zeta_5) &= (-c_2-c_6+c_7-c_8+c_{11}+c_{13})\zeta_3\zeta_4\zeta_5 - c_7\zeta_3^2\zeta_4 + c_6\zeta_3\zeta_4^2\\
&+ c_8\zeta_3^2\zeta_5 - c_{11}\zeta_3\zeta_5^2 - c_{13}\zeta_4^2\zeta_5 + c_2\zeta_4\zeta_5^2 + 3(-c_7+c_8)\zeta_3^3\zeta_4\zeta_5 + 3(c_6-c_{13})\zeta_3\zeta_4^3\zeta_5\\
&+ 3(c_2-c_{11})\zeta_3\zeta_4\zeta_5^3 + 3(-c_2+c_6-c_7-c_8+c_{11}+c_{13})\zeta_3^2\zeta_4^2\zeta_5\\
&+ 3(-c_2-c_6+c_7+c_8-c_{11}+c_{13})\zeta_3^2\zeta_4\zeta_5^2 + 3(c_2-c_6+c_7-c_8+c_{11}-c_{13})\zeta_3\zeta_4^2\zeta_5^2\\
&+ 3(-c_2+c_6-c_7-c_8+c_{11}+c_{13})\zeta_3^3\zeta_4^3\zeta_5 + 3(-c_2-c_6+c_7+c_8-c_{11}+c_{13})\zeta_3^3\zeta_4\zeta_5^3\\
&+ 3(c_2-c_6+c_7-c_8+c_{11}-c_{13})\zeta_3\zeta_4^3\zeta_5^3 + 6(-c_2+c_6-c_7+c_{11})\zeta_3^2\zeta_4^2\zeta_5^3\\
&+ 6(c_6-c_8+c_{11}-c_{13})\zeta_3^2\zeta_4^3\zeta_5^2 + 6(-c_2-c_7+c_8+c_{13})\zeta_3^3\zeta_4^2\zeta_5^2\\
&- 3c_7\zeta_3^4\zeta_4\zeta_5^2 + 3c_6\zeta_3\zeta_4^4\zeta_5^2 + 3c_8\zeta_3^4\zeta_4^2\zeta_5 - 3c_{11}\zeta_3\zeta_4^2\zeta_5^4 - 3c_{13}\zeta_3^2\zeta_4^4\zeta_5 + 3c_2\zeta_3^2\zeta_4\zeta_5^4\\
&+ 2(c_6-c_7)\zeta_3^4\zeta_4^4\zeta_5 + 2(c_8-c_{11})\zeta_3^4\zeta_4\zeta_5^4 + 2(c_2-c_{13})\zeta_3\zeta_4^4\zeta_5^4\\
&+ 2(-c_7+c_8)\zeta_3^5\zeta_4^2\zeta_5^2 + 2(c_6-c_{13})\zeta_3^2\zeta_4^5\zeta_5^2 + 2(c_2-c_{11})\zeta_3^2\zeta_4^2\zeta_5^5\\
&+ 3(c_2+c_6-c_7+c_8-c_{11}-c_{13})\zeta_3^3\zeta_4^3\zeta_5^3\\
&+ (-5c_2+3c_6-3c_7+3c_8+5c_{11}+5c_{13})\zeta_3^4\zeta_4^3\zeta_5^2 \qquad\qquad (\text{B.2.2})\\
&+ (-5c_2+3c_6-3c_7-5c_8+5c_{11}-3c_{13})\zeta_3^3\zeta_4^4\zeta_5^2\\
&+ (-5c_2-5c_6-3c_7+3c_8-3c_{11}+5c_{13})\zeta_3^4\zeta_4^2\zeta_5^3\\
&+ (3c_2+3c_6+5c_7-5c_8+5c_{11}-3c_{13})\zeta_3^2\zeta_4^4\zeta_5^3\\
&+ (3c_2-5c_6+5c_7+3c_8-3c_{11}+5c_{13})\zeta_3^3\zeta_4^2\zeta_5^4\\
&+ (3c_2-5c_6+5c_7-5c_8-3c_{11}-3c_{13})\zeta_3^2\zeta_4^3\zeta_5^4\\
&+ 3(-c_2-c_6-c_7+c_8+c_{11}+c_{13})\zeta_3^5\zeta_4^3\zeta_5^3 + 3(-c_2+c_6+c_7-c_8+c_{11}-c_{13})\zeta_3^3\zeta_4^5\zeta_5^3\\
&+ 3(c_2-c_6+c_7-c_8-c_{11}+c_{13})\zeta_3^3\zeta_4^3\zeta_5^5 + 6(-c_2+c_6-c_7+c_{11})\zeta_3^4\zeta_4^4\zeta_5^3\\
&+ 6(-c_6+c_8-c_{11}+c_{13})\zeta_3^4\zeta_4^3\zeta_5^4 + 6(c_2+c_7-c_8-c_{13})\zeta_3^3\zeta_4^4\zeta_5^4\\
&+ 3(c_6-c_7)\zeta_3^5\zeta_4^5\zeta_5^3 + 3(c_8-c_{11})\zeta_3^5\zeta_4^3\zeta_5^5 + 3(c_2-c_{13})\zeta_3^3\zeta_4^5\zeta_5^5\\
&+ 3(-c_2-c_6-c_7+c_8+c_{11}+c_{13})\zeta_3^5\zeta_4^4\zeta_5^4 + (-c_2+c_6+c_7-c_8+c_{11}-c_{13})\zeta_3^4\zeta_4^5\zeta_5^4\\
&+ 3(c_2-c_6+c_7-c_8-c_{11}+c_{13})\zeta_3^4\zeta_4^4\zeta_5^5 - c_7\zeta_3^6\zeta_4^5\zeta_5^4 + c_6\zeta_3^5\zeta_4^6\zeta_5^4\\
&+ c_8\zeta_3^6\zeta_4^4\zeta_5^5 - c_{11}\zeta_3^4\zeta_4^4\zeta_5^6 - c_{13}\zeta_3^4\zeta_4^6\zeta_5^5 + c_2\zeta_3^4\zeta_4^5\zeta_5^6\\
&+ (-c_2-c_6+c_7-c_8+c_{11}+c_{13})\zeta_3^5\zeta_4^5\zeta_5^5.
\end{aligned}
$$

B.3 α'-corrections to the gravitational planar amplitude

The α' expansion of the string graviton amplitude is given in Eq. (2.6.13). The ten-degree polynomial appearing in both the α'^5 and α'^7 terms is given by

$$\begin{aligned}
Q_{10}(\zeta_3,\zeta_4,\zeta_5) = {}& \zeta_4^2\zeta_5^4\zeta_3^4 - \zeta_4^3\zeta_5^3\zeta_3^4 + \zeta_4\zeta_5^3\zeta_3^4 + \zeta_4^2\zeta_3^4 + \zeta_4^4\zeta_5^2\zeta_3^4 - \zeta_4^2\zeta_5^2\zeta_3^4 \\
&+ \zeta_5^2\zeta_3^4 + \zeta_4^3\zeta_5\zeta_3^4 - \zeta_4\zeta_5\zeta_3^4 - \zeta_4^3\zeta_5^4\zeta_3^3 + \zeta_4\zeta_5^4\zeta_3^3 - \zeta_4^3\zeta_3^3 \\
&- \zeta_4^4\zeta_5^3\zeta_3^3 - \zeta_4^2\zeta_5^3\zeta_3^3 - \zeta_5^3\zeta_3^3 - \zeta_4^3\zeta_5^2\zeta_3^3 + \zeta_4\zeta_5^2\zeta_3^3 + \zeta_4\zeta_3^3 \\
&+ \zeta_4^4\zeta_5\zeta_3^3 + \zeta_4^2\zeta_5\zeta_3^3 + \zeta_5\zeta_3^3 + \zeta_4^4\zeta_3^2 + \zeta_4^4\zeta_5^4\zeta_3^2 - \zeta_4^2\zeta_5^4\zeta_3^2 \\
&+ \zeta_5^4\zeta_3^2 - \zeta_4^3\zeta_5^3\zeta_3^2 + \zeta_4\zeta_5^3\zeta_3^2 - \zeta_4^2\zeta_3^2 - \zeta_4^4\zeta_5^2\zeta_3^2 - 6\zeta_4^2\zeta_5^2\zeta_3^2 \\
&- \zeta_5^2\zeta_3^2 + \zeta_4^3\zeta_5\zeta_3^2 - \zeta_4\zeta_5\zeta_3^2 + \zeta_3^2 + \zeta_4^3\zeta_5^4\zeta_3 - \zeta_4\zeta_5^4\zeta_3 + \zeta_4^3\zeta_3 \\
&+ \zeta_4^4\zeta_5^3\zeta_3 + \zeta_4^2\zeta_5^3\zeta_3 + \zeta_5^3\zeta_3 + \zeta_4^3\zeta_5^2\zeta_3 - \zeta_4\zeta_5^2\zeta_3 - \zeta_4\zeta_3 \\
&- \zeta_4^4\zeta_5\zeta_3 - \zeta_4^2\zeta_5\zeta_3 - \zeta_5\zeta_3 + \zeta_4^2\zeta_5^4 - \zeta_4^3\zeta_5^3 + \zeta_4\zeta_5^3 \\
&+ \zeta_4^2 + \zeta_4^4\zeta_5^2 - \zeta_4^2\zeta_5^2 + \zeta_5^2 + \zeta_4^3\zeta_5 - \zeta_4\zeta_5.
\end{aligned} \tag{B.3.1}$$

It contains terms of degree 10, 8, 6, 4, and 2. The numerator associated with the α'^6 term is

$$\begin{aligned}
Q_{12}(\zeta_3,\zeta_4,\zeta_5) = {}& 2\zeta_4^4\zeta_5^4\zeta_3^4 + 3\zeta_4^3\zeta_5^3\zeta_3^4 + 3\zeta_4^2\zeta_5^2\zeta_3^4 + 3\zeta_4^3\zeta_5^4\zeta_3^3 + 3\zeta_4^4\zeta_5^3\zeta_3^3 + 3\zeta_4^2\zeta_5^3\zeta_3^3 \\
&+ 3\zeta_4^3\zeta_5^2\zeta_3^3 + 4\zeta_4\zeta_5^2\zeta_3^3 + 4\zeta_4^2\zeta_5\zeta_3^3 + 3\zeta_4^2\zeta_5^4\zeta_3^2 + 3\zeta_4^3\zeta_5^3\zeta_3^2 + 4\zeta_4\zeta_5^3\zeta_3^2 \\
&+ 3\zeta_4^2\zeta_3^2 + 3\zeta_4^4\zeta_5^2\zeta_3^2 - 2\zeta_4^2\zeta_5^2\zeta_3^2 + 3\zeta_5^2\zeta_3^2 + 4\zeta_4^3\zeta_5\zeta_3^2 + 3\zeta_4\zeta_5\zeta_3^2 \\
&+ 4\zeta_4^2\zeta_5^3\zeta_3 + 4\zeta_4^3\zeta_5^2\zeta_3 + 3\zeta_4\zeta_5^2\zeta_3 + 3\zeta_4\zeta_3 + 3\zeta_4^2\zeta_5\zeta_3 + 3\zeta_5\zeta_3 \\
&+ 3\zeta_4^2\zeta_5^2 + 3\zeta_4\zeta_5 + 2.
\end{aligned} \tag{B.3.2}$$

This is a degree 12 polynomial including monomials of degree 12, 10, 8, 6, 4, 2, and 0. Finally, the numerator determining the α'^8 corrections is the following nonhomogeneous degree 22 polynomial

$$\begin{aligned}
Q_{22}(\zeta_3,\zeta_4,\zeta_5) = {}& 8Q_{10}Q_{12} + 3(1+\zeta_3\zeta_4)^2(1+\zeta_3\zeta_5)^2(1+\zeta_4\zeta_5)^2\Big(2\zeta_4^2\zeta_3^4 + 2\zeta_5^2\zeta_3^4 \\
&- 2\zeta_4\zeta_5\zeta_3^4 - 2\zeta_4^3\zeta_3^3 - 2\zeta_5^3\zeta_3^3 - \zeta_4\zeta_5^2\zeta_3^3 - \zeta_4^2\zeta_5\zeta_3^3 + 2\zeta_4^4\zeta_3^2 + 2\zeta_5^4\zeta_3^2 \\
&- \zeta_4\zeta_5^3\zeta_3^2 + 6\zeta_4^2\zeta_5^2\zeta_3^2 - \zeta_4^3\zeta_5\zeta_3^2 - 2\zeta_4\zeta_5^4\zeta_3 - \zeta_4^2\zeta_5^3\zeta_3 - \zeta_4^3\zeta_5^2\zeta_3 \\
&- 2\zeta_4^4\zeta_5\zeta_3 + 2\zeta_4^2\zeta_5^4 - 2\zeta_4^3\zeta_5^3 + 2\zeta_4^4\zeta_5^2\Big),
\end{aligned} \tag{B.3.3}$$

where Q_{10} and Q_{12} are the polynomials given in Eqs. (B.3.1) and (B.3.2).

B.4 Sudakov representation of general n-point process

For any n-point process, the following notation is considered (see Fig. 3.5):

$$p + q \quad \longrightarrow \quad p' + q' + k_1 + k_2 + \ldots + k_{n-4} \,, \tag{B.4.1}$$

where $k_i := q_i - q_{i+1}$ and each of the internal momentum vectors is parametrized as

$$q_i \equiv \alpha_i p + \beta_i q + \mathbf{q_i} \qquad \text{with} \qquad \mathbf{q_i} = q_i^{\perp}(0, \cos\theta_i, \sin\theta_i, 0) \,. \tag{B.4.2}$$

Analogously to the derivations of the $n = 4, 5$ and 6 cases, the energies of the outgoing particles have this form

$$\begin{aligned}
\omega_{p'} &= \frac{\sqrt{s}}{2}(1 - \alpha_1 - \beta_1) = -\frac{\sqrt{s}}{2}\beta_1\left(1 + \sigma_{p'}\sigma^*_{p'}\right) \,, \\
\omega_{k_i} &= \frac{\sqrt{s}}{2}\left(\alpha_i + \beta_i - \alpha_{i+1} - \beta_{i+1}\right) = \frac{\sqrt{s}}{2}(\beta_i - \beta_{i+1})\left(1 + \sigma_{k_i}\sigma^*_{k_i}\right) \,, \\
\omega_{q'} &= \frac{\sqrt{s}}{2}(1 + \alpha_{n-3} + \beta_{n-3}) = \frac{\sqrt{s}}{2}(1 + \beta_{n-3})\left(1 + \sigma_{q'}\sigma^*_{q'}\right) \,;
\end{aligned} \tag{B.4.3}$$

unit vectors point towards the following directions

$$\begin{aligned}
\mathbf{u}_{p'} &= \left(\frac{-2q_1^{\perp}\cos\theta_1}{\sqrt{s}\,(1 - \alpha_1 - \beta_1)}, \frac{-2q_1^{\perp}\sin\theta_1}{\sqrt{s}\,(1 - \alpha_1 - \beta_1)}, \frac{1 - \alpha_1 + \beta_1}{1 - \alpha_1 - \beta_1}\right) \,, \\
\mathbf{u}_{k_i} &= \left(\frac{2\left(q_i^{\perp}\cos\theta_i - q_{i+1}^{\perp}\cos\theta_{i+1}\right)}{\sqrt{s}\,(\alpha_i + \beta_i - \alpha_{i+1} - \beta_{i+1})}, \frac{2\left(q_i^{\perp}\sin\theta_i - q_{i+1}^{\perp}\sin\theta_{i+1}\right)}{\sqrt{s}\,(\alpha_i + \beta_i - \alpha_{i+1} - \beta_{i+1})}, \frac{\alpha_i - \beta_i - \alpha_{i+1} + \beta_{i+1}}{\alpha_i + \beta_i - \alpha_{i+1} - \beta_{i+1}}\right) \,, \\
\mathbf{u}_{q'} &= \left(\frac{2q_{n-3}^{\perp}\cos\theta_{n-3}}{\sqrt{s}\,(1 + \alpha_{n-3} + \beta_{n-3})}, \frac{2q_{n-3}^{\perp}\sin\theta_{n-3}}{\sqrt{s}\,(1 + \alpha_{n-3} + \beta_{n-3})}, \frac{\alpha_{n-3} - \beta_{n-3} - 1}{1 + \alpha_{n-3} + \beta_{n-3}}\right) \,;
\end{aligned} \tag{B.4.4}$$

and onshellness can be expressed in terms of the transverse momentum variables as

$$\begin{aligned}
(p')^2 = 0 &\quad\Rightarrow\quad |\hat{Q}_1|^2 = (\alpha_1 - 1)\beta_1 \,, \\
k_i^2 = 0 &\quad\Rightarrow\quad |\hat{Q}_i - \hat{Q}_{i+1}|^2 = (\alpha_i - \alpha_{i+1})(\beta_i - \beta_{i+1}) \,, \\
(q')^2 = 0 &\quad\Rightarrow\quad |\hat{Q}_{n-3}|^2 = \alpha_{n-3}(1 + \beta_{n-3}) \,.
\end{aligned} \tag{B.4.5}$$

Fairlie's solution acquire also a pretty compact form

$$\sigma_{p'} = -\frac{\hat{Q}_1}{\beta_1} \,, \qquad \sigma_{k_i} = \frac{\hat{Q}_i - \hat{Q}_{i+1}}{\beta_i - \beta_{i+1}} \,, \qquad \sigma_{q'} = \frac{\hat{Q}_{n-3}}{1 + \beta_{n-3}} \,, \tag{B.4.6}$$

where each of the punctures, by means of the free azimuthal angles, projects onto the equatorial plane on circumferences of radii

$$R_{p'} = 2\sqrt{\frac{|\hat{Q}_1|^2}{(1-\alpha_1-\beta_1)^2}} = 2\sqrt{\frac{(\alpha_1-1)\beta_1}{(1-\alpha_1-\beta_1)^2}} \ ,$$

$$R_{k_i} = 2\sqrt{\frac{|\hat{Q}_i-\hat{Q}_{i+1}|^2}{(\alpha_i+\beta_i-\alpha_{i+1}-\beta_{i+1})^2}} = 2\sqrt{\frac{(\alpha_i-\alpha_{i+1})(\beta_i-\beta_{i+1})}{(\alpha_i+\beta_i-\alpha_{i+1}-\beta_{i+1})^2}} \ ,$$

$$R_{q'} = 2\sqrt{\frac{|\hat{Q}_{n-3}|^2}{(1+\alpha_{n-3}+\beta_{n-3})^2}} = 2\sqrt{\frac{\alpha_{n-3}(1+\beta_{n-3})}{(1+\alpha_{n-3}+\beta_{n-3})^2}} \ . \tag{B.4.7}$$

B.5 $\hat{Q}_{13}^2$ on-shellness coefficients

This is the Sudakov dependence of the coefficients constraining $|\hat{Q}_{13}|^2$ in Eq. (3.7.12) from the on-shell conditions in a 6-point scattering process:

$$c_1 = -\alpha_3+\beta_1-2\alpha_1\beta_1+\alpha_2\beta_1+\alpha_1\beta_2-2\alpha_2\beta_2+\alpha_3\beta_2+\alpha_2\beta_3-2\alpha_3\beta_3 \ , \tag{B.5.1}$$

$$c_2 = \left((-1+\alpha_2)\beta_1+(\alpha_1-\alpha_2)\beta_2\right)\left(\alpha_3-\alpha_2\beta_2+\alpha_3\beta_2+\alpha_2\beta_3\right) \ , \tag{B.5.2}$$

$$c_3 = \left(\alpha_3+\beta_1-\alpha_1\beta_1+\alpha_3\beta_3\right)\left(\alpha_1(-\beta_1+\beta_2)+\alpha_2(\beta_1-\beta_3)+\alpha_3(-\beta_2+\beta_3)\right) \ , \tag{B.5.3}$$

$$\begin{aligned} c_4 = &-(\alpha_3+\beta_1-\alpha_2\beta_1-\alpha_1\beta_2+\alpha_3\beta_2+\alpha_2\beta_3)\times \\ &\Bigg[\alpha_3\beta_1(\beta_2-\beta_3)+\alpha_2\Bigg(\alpha_3(\beta_1-\beta_2)(1+\beta_3)+\beta_1(-\beta_2+\beta_3)\Bigg)+ \\ &\alpha_1\Bigg(\alpha_2\beta_1(\beta_2-\beta_3)+\alpha_3(\beta_2-\beta_1(1+\beta_2)+\beta_2\beta_3)\Bigg)\Bigg] \ . \end{aligned} \tag{B.5.4}$$

B.6 Scattering Equations $n = 6$

The full set of 6-point SE in their standard form, after the partial fixing $\sigma_p \to \infty$ and $\sigma_q \to 0$, are written in Sudakov parametrization in the following way

$$\frac{1-\alpha_1}{\sigma_{p'}} + \frac{\alpha_1-\alpha_2}{\sigma_{k_1}} + \frac{\sigma_{q'}(\alpha_2-\alpha_3)+\sigma_{k_2}\alpha_3}{\sigma_{q'}\sigma_{k_2}} = 0 \ ,$$

$$\begin{aligned} &\frac{-1+\alpha_1}{\sigma_{p'}} - \frac{(\hat{q}_2^{\perp})^2+\beta_2-\alpha_2\beta_2}{\sigma_{p'}-\sigma_{k_1}} + \frac{-\hat{Q}_{13}^2+(-1+\alpha_1-\alpha_3)(-1+\beta_1-\beta_3)}{\sigma_{p'}-\sigma_{q'}} + \\ &+\frac{\hat{Q}_{13}^2+(\hat{q}_2^{\perp})^2-\alpha_1\beta_1+\beta_2-\alpha_2\beta_2+\alpha_3(-1+\beta_1-\beta_3)+(-1+\alpha_1)\beta_3}{\sigma_{p'}-\sigma_{k_2}} = 0 \ , \end{aligned}$$

$$
\begin{aligned}
&-\frac{\alpha_3}{\sigma_{q'}}+\frac{-(\hat{q}_2^{\perp})^2+\alpha_2+\alpha_2\beta_2}{\sigma_{q'}-\sigma_{k_2}}+\frac{\hat{Q}_{13}^2-(-1+\alpha_1-\alpha_3)(-1+\beta_1-\beta_3)}{\sigma_{p'}-\sigma_{q'}}+\\
&+\frac{\hat{Q}_{13}^2+(\hat{q}_2^{\perp})^2+\alpha_1+\beta_1-\alpha_1\beta_1+\alpha_3\beta_1-\alpha_2(1+\beta_2)+\alpha_1\beta_3-\alpha_3\beta_3}{\sigma_{q'}-\sigma_{k_1}}=0\ ,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{-\alpha_1+\alpha_2}{\sigma_{k_1}}+\frac{(\hat{q}_2^{\perp})^2+\beta_2-\alpha_2\beta_2}{\sigma_{p'}-\sigma_{k_1}}+\frac{-\hat{Q}_{13}^2+(\alpha_1-\alpha_3)(\beta_1-\beta_3)}{\sigma_{k_1}-\sigma_{k_2}}-\\
&-\frac{\hat{Q}_{13}^2+(\hat{q}_2^{\perp})^2+\alpha_1+\beta_1-\alpha_1\beta_1+\alpha_3\beta_1-\alpha_2(1+\beta_2)+\alpha_1\beta_3-\alpha_3\beta_3}{\sigma_{q'}-\sigma_{k_1}}=0\ ,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{-\alpha_2+\alpha_3}{\sigma_{k_2}}+\frac{(\hat{q}_2^{\perp})^2-\alpha_2-\alpha_2\beta_2}{\sigma_{q'}-\sigma_{k_2}}+\frac{\hat{Q}_{13}^2-(\alpha_1-\alpha_3)(\beta_1-\beta_3)}{\sigma_{k_1}-\sigma_{k_2}}-\\
&-\frac{\hat{Q}_{13}^2+(\hat{q}_2^{\perp})^2-\alpha_1\beta_1+\beta_2-\alpha_2\beta_2+\alpha_3(-1+\beta_1-\beta_3)+(-1+\alpha_1)\beta_3}{\sigma_{p'}-\sigma_{k_2}}=0\ . \qquad \text{(B.6.1)}
\end{aligned}
$$

The same 6-point SE in the MRK regime read

$$
\begin{aligned}
\frac{1}{\sigma_{p'}}+\frac{\alpha_1}{\sigma_{k_1}}+\frac{\alpha_2}{\sigma_{k_2}}+\frac{\alpha_3}{\sigma_{q'}}&=0\ ,\\
\frac{-1}{\sigma_{p'}}+\frac{-\beta_2}{\sigma_{p'}-\sigma_{k_1}}+\frac{1}{\sigma_{p'}-\sigma_{q'}}+\frac{-\beta_3}{\sigma_{p'}-\sigma_{k_2}}&=0\ ,\\
\frac{-\alpha_3}{\sigma_{q'}}+\frac{\alpha_2}{\sigma_{q'}-\sigma_{k_2}}+\frac{1}{\sigma_{q'}-\sigma_{p'}}+\frac{\alpha_1}{\sigma_{q'}-\sigma_{k_1}}&=0\ ,\\
\frac{-\alpha_1}{\sigma_{k_1}}+\frac{-\beta_2}{\sigma_{k_1}-\sigma_{p'}}+\frac{-\alpha_1\beta_3}{\sigma_{k_1}-\sigma_{k_2}}+\frac{\alpha_1}{\sigma_{k_1}-\sigma_{q'}}&=0\ ,\\
\frac{-\alpha_2}{\sigma_{k_2}}+\frac{\alpha_2}{\sigma_{k_2}-\sigma_{q'}}+\frac{-\alpha_1\beta_3}{\sigma_{k_2}-\sigma_{k_1}}+\frac{-\beta_3}{\sigma_{k_2}-\sigma_{p'}}&=0\ , \qquad \text{(B.6.2)}
\end{aligned}
$$

whereas their polynomial form in the MRK limit is

$$
\begin{aligned}
\sigma_{p'}\beta_1-\sigma_{q'}+\sigma_{k_1}\beta_2+\sigma_{k_2}\beta_3&=0\ ,\\
\sigma_{p'}\sigma_{q'}\alpha_1+\sigma_{p'}\sigma_{k_1}(q_2^{\perp})^2-\sigma_{p'}\sigma_{k_2}\alpha_1\beta_3+\sigma_{q'}\sigma_{k_1}&=0\ ,\\
\sigma_{p'}\sigma_{q'}\sigma_{k_1}\alpha_2+\sigma_{p'}\sigma_{q'}\sigma_{k_2}\alpha_1+\sigma_{p'}\sigma_{k_1}\sigma_{k_2}\alpha_3+\sigma_{q'}\sigma_{k_1}\sigma_{k_2}&=0\ . \qquad \text{(B.6.3)}
\end{aligned}
$$

Bibliography

[1] D. Medrano Jiménez, A. Sabio Vera and M. Á. Vázquez-Mozo, *Planar Zeros in Gauge Theories and Gravity*, *JHEP* **09** (2016) 006 [1607.04605].

[2] D. Medrano Jiménez, A. Sabio Vera and M. Á. Vázquez-Mozo, *Projectivity of Planar Zeros in Field and String Theory Amplitudes*, *JHEP* **05** (2017) 011 [1703.07274].

[3] G. Chachamis, D. Medrano Jiménez, A. Sabio Vera and M. Á. Vázquez-Mozo, *Sudakov Representation of the Cachazo-He-Yuan Scattering Equations Formalism*, *JHEP* **01** (2018) 057 [1712.04288].

[4] J. M. Henn and J. C. Plefka, *Scattering Amplitudes in Gauge Theories*, *Lect. Notes Phys.* **883** (2014) pp.1.

[5] H. Elvang and Y.-t. Huang, *Scattering Amplitudes in Gauge Theory and Gravity.* Cambridge University Press, 2015.

[6] L. J. Dixon, *A brief introduction to modern amplitude methods*, in *Proceedings, 2012 European School of High-Energy Physics (ESHEP 2012): La Pommeraye, Anjou, France, June 06-19, 2012*, pp. 31–67, 2014, 1310.5353, DOI.

[7] T. R. Taylor, *A Course in Amplitudes*, *Phys. Rept.* **691** (2017) 1 [1703.05670].

[8] S. Weinzierl, *Tales of 1001 Gluons*, *Phys. Rept.* **676** (2017) 1 [1610.05318].

[9] M. L. Mangano and S. J. Parke, *Multiparton amplitudes in gauge theories*, *Phys. Rept.* **200** (1991) 301 [hep-th/0509223].

[10] R. Kleiss and H. Kuijf, *Multi - Gluon Cross-sections and Five Jet Production at Hadron Colliders*, *Nucl. Phys.* **B312** (1989) 616.

[11] V. Del Duca, L. J. Dixon and F. Maltoni, *New color decompositions for gauge amplitudes at tree and loop level*, *Nucl. Phys.* **B571** (2000) 51 [hep-ph/9910563].

[12] Z. Bern, J. J. M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev.* **D78** (2008) 085011 [0805.3993].

[13] S. J. Parke and T. R. Taylor, *An Amplitude for n Gluon Scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.

[14] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052].

[15] Z. Bern, J. J. M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [1004.0476].

[16] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, *Phys. Rev.* **D82** (2010) 065003 [1004.0693].

[17] H. Kawai, D. C. Lewellen and S. H. H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys.* **B269** (1986) 1.

[18] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, *The Momentum Kernel of Gauge and Gravity Theories*, *JHEP* **01** (2011) 001 [1010.3933].

[19] S. Stieberger and T. R. Taylor, *Graviton as a Pair of Collinear Gauge Bosons*, *Phys. Lett.* **B739** (2014) 457 [1409.4771].

[20] S. Stieberger and T. R. Taylor, *Graviton Amplitudes from Collinear Limits of Gauge Amplitudes*, *Phys. Lett.* **B744** (2015) 160 [1502.00655].

[21] A. Sabio Vera and M. Á. Vázquez-Mozo, *The Double Copy Structure of Soft Gravitons*, *JHEP* **03** (2015) 070 [1412.3699].

[22] R. W. Brown, *Understanding something about nothing: Radiation zeros*, *AIP Conf. Proc.* **350** (1995) 261 [hep-th/9506018].

[23] T. Han, *Exact and approximate radiation amplitude zeros: Phenomenological aspects*, *AIP Conf. Proc.* **350** (1995) 224 [hep-ph/9506286].

[24] U. Baur and R. W. Brown, *Zero zeros after all these (20) years*, in *Workshop on the Transition from Low to High Q Form-factors (To Honor the Occasion of the 60th Birthday of Stanley Brodsky) Athens, Georgia, September 17, 1999*, 1999, hep-ph/9909522.

[25] R. W. Brown, D. Sahdev and K. O. Mikaelian, *$W^{\pm}Z^0$ and $W^{\pm}\gamma$ Pair Production in νe, pp, and anti-pp Collisions*, *Phys. Rev.* **D20** (1979) 1164.

[26] K. O. Mikaelian, M. A. Samuel and D. Sahdev, *The Magnetic Moment of Weak Bosons Produced in pp and $p\bar{p}$ Collisions*, *Phys. Rev. Lett.* **43** (1979) 746.

[27] D0 collaboration, V. M. Abazov et al., *First study of the radiation-amplitude zero in $W\gamma$ production and limits on anomalous $WW\gamma$ couplings at $\sqrt{s}$ = 1.96- TeV*, *Phys. Rev. Lett.* **100** (2008) 241805 [0803.0030].

[28] CMS collaboration, S. Chatrchyan et al., *Measurement of the $W\gamma$ and $Z\gamma$ inclusive cross sections in pp collisions at $\sqrt{s} = 7\,TeV$ and limits on anomalous triple gauge boson couplings*, *Phys. Rev.* **D89** (2014) 092005 [1308.6832].

[29] G. Passarino, *Radiation zeros and gravity*, *Nucl. Phys.* **B241** (1984) 48.

[30] M. Heyssler and W. J. Stirling, *Radiation zeros at HERA: More about nothing*, *Eur. Phys. J.* **C4** (1998) 289 [hep-ph/9707373].

[31] M. Heyssler and W. J. Stirling, *Radiation zeros in high-energy e^+e^- annihilation into hadrons*, *Eur. Phys. J.* **C5** (1998) 475 [hep-ph/9712314].

[32] I. Rodriguez and O. A. Sampayo, *Tau anomalous couplings and radiation zeros in the $e^+e^- \to \tau\bar{\tau}\gamma$ process*, hep-ph/0312316.

[33] L. A. Harland-Lang, *Planar radiation zeros in five-parton QCD amplitudes*, *JHEP* **05** (2015) 146 [1503.06798].

[34] B. Kol and R. Shir, *Color structures and permutations*, *JHEP* **11** (2014) 020 [1403.6837].

[35] D. H. Sattinger and O. L. Weaver, *Lie Groups and Algebras with Applications to Physics, Geometry and Mechanics*. Springer-Verlag New York, 1986, 10.1007/978-1-4757-1910-9.

[36] R. N. Mohapatra, *Unification and Supersymmetry: The Frontiers of Quark-Lepton Physics*. Springer-Verlag New York, 2003, 10.1007/b98865.

[37] F. A. Berends, W. T. Giele and H. Kuijf, *On relations between multi-gluon and multi-graviton scattering*, *Phys. Lett.* **B211** (1988) 91.

[38] T. He, P. Mitra and A. Strominger, *2D Kac-Moody Symmetry of 4D Yang-Mills Theory*, *JHEP* **10** (2016) 137 [1503.02663].

[39] Y.-t. Huang and H. Johansson, *Equivalent D=3 Supergravity Amplitudes from Double Copies of Three-Algebra and Two-Algebra Gauge Theories*, *Phys. Rev. Lett.* **110** (2013) 171601 [1210.2255].

[40] C. R. Mafra, O. Schlotterer and S. Stieberger, *Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation*, *Nucl. Phys.* **B873** (2013) 419 [1106.2645].

[41] C. R. Mafra, O. Schlotterer and S. Stieberger, *Complete N-Point Superstring Disk Amplitude II. Amplitude and Hypergeometric Function Structure*, *Nucl. Phys.* **B873** (2013) 461 [1106.2646].

[42] S. Stieberger and T. R. Taylor, *Closed String Amplitudes as Single-Valued Open String Amplitudes*, *Nucl. Phys.* **B881** (2014) 269 [1401.1218].

[43] S. Stieberger, *Open & Closed vs. Pure Open String Disk Amplitudes*, 0907.2211.

[44] S. Stieberger, *Closed superstring amplitudes, single-valued multiple zeta values and the Deligne associator*, *J. Phys.* **A47** (2014) 155401 [1310.3259].

[45] R. Monteiro, D. O'Connell and C. D. White, *Black holes and the double copy*, *JHEP* **12** (2014) 056 [1410.0239].

[46] A. Sabio Vera, E. Serna Campillo and M. Á. Vázquez-Mozo, *Color-Kinematics Duality and the Regge Limit of Inelastic Amplitudes*, *JHEP* **04** (2013) 086 [1212.5103].

[47] A. D. Polyanin and A. V. Manzhirov, *Handbook of Mathematics for Engineers and Scientists*. Chapman & Hall / CRC, 2006.

[48] H. Johansson, A. Sabio Vera, E. Serna Campillo and M. Á. Vázquez-Mozo, *Color-Kinematics Duality in Multi-Regge Kinematics and Dimensional Reduction*, *JHEP* **10** (2013) 215 [1307.3106].

[49] N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, *Minimal Basis for Gauge Theory Amplitudes*, *Phys. Rev. Lett.* **103** (2009) 161602 [0907.1425].

[50] J. Broedel, O. Schlotterer and S. Stieberger, *Polylogarithms, Multiple Zeta Values and Superstring Amplitudes*, *Fortsch. Phys.* **61** (2013) 812 [1304.7267].

[51] A. Sabio Vera, E. Serna Campillo and M. Á. Vázquez-Mozo, *Graviton emission in Einstein-Hilbert gravity*, *JHEP* **03** (2012) 005 [1112.4494].

[52] M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, *Scattering amplitudes in $\mathcal{N} = 2$ Maxwell-Einstein and Yang-Mills/Einstein supergravity*, *JHEP* **01** (2015) 081 [1408.0764].

[53] L. J. Mason and D. Skinner, *Gravity, Twistors and the MHV Formalism*, *Commun. Math. Phys.* **294** (2010) 827 [0808.3907].

[54] D. Nguyen, M. Spradlin, A. Volovich and C. Wen, *The Tree Formula for MHV Graviton Amplitudes*, *JHEP* **07** (2010) 045 [0907.2276].

[55] F. C. S. Brown, *Polylogarithmes multiples uniformes en une variable*, *Compt. Rend. Math.* **338** (2004) 527.

[56] P. Di Vecchia, R. Marotta and M. Mojaza, *Soft theorem for the graviton, dilaton and the Kalb-Ramond field in the bosonic string*, *JHEP* **05** (2015) 137 [1502.05258].

[57] A. Sen, *Soft Theorems in Superstring Theory*, *JHEP* **06** (2017) 113 [1702.03934].

[58] T. He, P. Mitra, A. P. Porfyriadis and A. Strominger, *New Symmetries of Massless QED*, *JHEP* **10** (2014) 112 [1407.3789].

[59] V. Lysov, S. Pasterski and A. Strominger, *Lows Subleading Soft Theorem as a Symmetry of QED*, *Phys. Rev. Lett.* **113** (2014) 111601 [1407.3814].

[60] D. Kapec, M. Pate and A. Strominger, *New Symmetries of QED*, *Adv. Theor. Math. Phys.* **21** (2017) 1769 [1506.02906].

[61] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, 1703.05448.

[62] F. Cachazo, S. He and E. Y. Yuan, *Scattering equations and Kawai-Lewellen-Tye orthogonality*, *Phys. Rev.* **D90** (2014) 065001 [1306.6575].

[63] F. Cachazo, S. He and E. Y. Yuan, *Scattering of Massless Particles in Arbitrary Dimensions*, *Phys. Rev. Lett.* **113** (2014) 171601 [1307.2199].

[64] F. Cachazo, S. He and E. Y. Yuan, *Scattering of Massless Particles: Scalars, Gluons and Gravitons*, *JHEP* **07** (2014) 033 [1309.0885].

[65] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily and P. H. Damgaard, *Integration Rules for Scattering Equations*, *JHEP* **09** (2015) 129 [1506.06137].

[66] S. Weinzierl, *On the solutions of the scattering equations*, *JHEP* **04** (2014) 092 [1402.2516].

[67] C. Kalousios, *Massless scattering at special kinematics as Jacobi polynomials*, *J. Phys.* **A47** (2014) 215402 [1312.7743].

[68] C. Duhr and Z. Liu, *Multi-Regge kinematics and the scattering equations*, *JHEP* **01** (2019) 146 [1811.06478].

[69] J. A. Farrow, *A Monte Carlo Approach to the 4D Scattering Equations*, *JHEP* **08** (2018) 085 [1806.02732].

[70] Z. Liu and X. Zhao, *Bootstrapping solutions of scattering equations*, *JHEP* **02** (2019) 071 [1810.00384].

[71] D. J. Gross and P. F. Mende, *The High-Energy Behavior of String Scattering Amplitudes*, *Phys. Lett.* **B197** (1987) 129.

[72] D. J. Gross and P. F. Mende, *String Theory Beyond the Planck Scale*, *Nucl. Phys.* **B303** (1988) 407.

[73] D. J. Gross and J. L. Manes, *The High-energy Behavior of Open String Scattering*, *Nucl. Phys.* **B326** (1989) 73.

[74] D. B. Fairlie and D. E. Roberts, *Dual Models without Tachyons - A new Approach*, .

[75] F. Cachazo, *Fundamental BCJ Relation in N=4 SYM From The Connected Formulation*, 1206.5970.

[76] T. Adamo, E. Casali and D. Skinner, *Ambitwistor strings and the scattering equations at one loop*, *JHEP* **04** (2014) 104 [1312.3828].

[77] Y. Geyer, L. Mason, R. Monteiro and P. Tourkine, *Loop Integrands for Scattering Amplitudes from the Riemann Sphere*, *Phys. Rev. Lett.* **115** (2015) 121603 [1507.00321].

[78] F. Cachazo, S. He and E. Y. Yuan, *One-Loop Corrections from Higher Dimensional Tree Amplitudes*, *JHEP* **08** (2016) 008 [1512.05001].

[79] Y. Geyer, L. Mason, R. Monteiro and P. Tourkine, *One-loop amplitudes on the Riemann sphere*, *JHEP* **03** (2016) 114 [1511.06315].

[80] Y. Geyer, A. E. Lipstein and L. J. Mason, *Ambitwistor Strings in Four Dimensions*, *Phys. Rev. Lett.* **113** (2014) 081602 [1404.6219].

[81] L. Mason and D. Skinner, *Ambitwistor strings and the scattering equations*, *JHEP* **07** (2014) 048 [1311.2564].

[82] V. V. Sudakov, *Vertex parts at very high-energies in quantum electrodynamics*, *Sov. Phys. JETP* **3** (1956) 65.

[83] D. B. Fairlie, *A Coding of Real Null Four-Momenta into World-Sheet Coordinates*, *Adv. Math. Phys.* **2009** (2009) 284689 [0805.2263].

[84] L. Dolan and P. Goddard, *The Polynomial Form of the Scattering Equations*, *JHEP* **07** (2014) 029 [1402.7374].

[85] C. Cardona and C. Kalousios, *Elimination and recursions in the scattering equations*, *Phys. Lett.* **B756** (2016) 180 [1511.05915].

[86] Y.-j. Du, F. Teng and Y.-s. Wu, *CHY formula and MHV amplitudes*, *JHEP* **05** (2016) 086 [1603.08158].

[87] P. D. B. Collins, *An Introduction to Regge Theory and High-Energy Physics*, Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge, UK, 2009, 10.1017/CBO9780511897603.

[88] Z. Liu, *Gravitational Scattering in the High-Energy Limit*, *JHEP* **02** (2019) 112 [1811.11710].

[89] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [hep-th/0312171].

[90] R. Roiban, M. Spradlin and A. Volovich, *On the tree level S matrix of Yang-Mills theory*, *Phys. Rev.* **D70** (2004) 026009 [hep-th/0403190].

[91] F. Cachazo and Y. Geyer, *A 'Twistor String' Inspired Formula For Tree-Level Scattering Amplitudes in N=8 SUGRA*, 1206.6511.

[92] F. Cachazo and D. Skinner, *Gravity from Rational Curves in Twistor Space*, *Phys. Rev. Lett.* **110** (2013) 161301 [1207.0741].

[93] S. G. Naculich, *Scattering equations and BCJ relations for gauge and gravitational amplitudes with massive scalar particles*, *JHEP* **09** (2014) 029 [1407.7836].

[94] F. Cachazo, S. He and E. Y. Yuan, *Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM*, *JHEP* **07** (2015) 149 [1412.3479].

[95] F. Cachazo, S. He and E. Y. Yuan, *Einstein-Yang-Mills Scattering Amplitudes From Scattering Equations*, *JHEP* **01** (2015) 121 [1409.8256].

[96] L. Dolan and P. Goddard, *Proof of the Formula of Cachazo, He and Yuan for Yang-Mills Tree Amplitudes in Arbitrary Dimension*, *JHEP* **05** (2014) 010 [1311.5200].

[97] F. Cachazo, S. He and E. Y. Yuan, *Scattering in Three Dimensions from Rational Maps*, *JHEP* **10** (2013) 141 [1306.2962].

[98] C. S. Lam, *Permutation Symmetry of the Scattering Equations*, *Phys. Rev.* **D91** (2015) 045019 [1410.8184].

[99] C. Kalousios, *Scattering equations, generating functions and all massless five point tree amplitudes*, *JHEP* **05** (2015) 054 [1502.07711].

[100] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily and P. H. Damgaard, *Scattering Equations and Feynman Diagrams*, *JHEP* **09** (2015) 136 [1507.00997].

[101] L. Dolan and P. Goddard, *General Solution of the Scattering Equations*, *JHEP* **10** (2016) 149 [1511.09441].

[102] N. Berkovits, *Infinite Tension Limit of the Pure Spinor Superstring*, *JHEP* **03** (2014) 017 [1311.4156].

[103] K. Ohmori, *Worldsheet Geometries of Ambitwistor String*, *JHEP* **06** (2015) 075 [1504.02675].

[104] E. Casali, Y. Geyer, L. Mason, R. Monteiro and K. A. Roehrig, *New Ambitwistor String Theories*, *JHEP* **11** (2015) 038 [1506.08771].

[105] T. Adamo and E. Casali, *Scattering equations, supergravity integrands, and pure spinors*, *JHEP* **05** (2015) 120 [1502.06826].

[106] N. E. J. Bjerrum-Bohr, P. H. Damgaard, P. Tourkine and P. Vanhove, *Scattering Equations and String Theory Amplitudes*, *Phys. Rev.* **D90** (2014) 106002 [1403.4553].

[107] H. Gomez and E. Y. Yuan, *N-point tree-level scattering amplitude in the new Berkovits' string*, *JHEP* **04** (2014) 046 [1312.5485].

[108] E. Casali and P. Tourkine, *Infrared behaviour of the one-loop scattering equations and supergravity integrands*, *JHEP* **04** (2015) 013 [1412.3787].

[109] C. González-Ballestero, L. Robledo and G. Bertsch, *Numeric and symbolic evaluation of the pfaffian of general skew-symmetric matrices*, *Computer Physics Communications* **182** (2011) 2213 .

[110] C. Cheung, K. Kampf, J. Novotny and J. Trnka, *Effective Field Theories from Soft Limits of Scattering Amplitudes*, *Phys. Rev. Lett.* **114** (2015) 221602 [1412.4095].

[111] K. Hinterbichler and A. Joyce, *Hidden symmetry of the Galileon*, *Phys. Rev.* **D92** (2015) 023503 [1501.07600].

[112] J. A. Cronin, *Phenomenological model of strong and weak interactions in chiral U(3) x U(3)*, *Phys. Rev.* **161** (1967) 1483.

[113] S. Weinberg, *Dynamical approach to current algebra*, *Phys. Rev. Lett.* **18** (1967) 188.

[114] S. Weinberg, *Nonlinear realizations of chiral symmetry*, *Phys. Rev.* **166** (1968) 1568.

[115] A. A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, hep-th/9908105.

[116] Y. Yuan, *Scattering Equations & S-Matrices*, Ph.D. thesis, U. Waterloo (main), 2015.

[117] Y.-J. Du and F. Teng, *BCJ numerators from reduced Pfaffian*, *JHEP* **04** (2017) 033 [1703.05717].

[118] R. Monteiro and D. O'Connell, *The Kinematic Algebras from the Scattering Equations*, *JHEP* **03** (2014) 110 [1311.1151].

[119] H. Johansson, A. Sabio Vera, E. Serna Campillo and M. Á. Vázquez-Mozo, *Color-kinematics duality and dimensional reduction for graviton emission in Regge limit*, in *International Workshop on Low X Physics (Israel 2013) Eilat, Israel, May 30-June 04, 2013*, 2013, 1310.1680.

[120] B. U. W. Schwab and A. Volovich, *Subleading Soft Theorem in Arbitrary Dimensions from Scattering Equations*, *Phys. Rev. Lett.* **113** (2014) 101601 [1404.7749].

[121] S. He, Y.-t. Huang and C. Wen, *Loop Corrections to Soft Theorems in Gauge Theories and Gravity*, *JHEP* **12** (2014) 115 [1405.1410].

[122] M. Bianchi, S. He, Y.-t. Huang and C. Wen, *More on Soft Theorems: Trees, Loops and Strings*, *Phys. Rev.* **D92** (2015) 065022 [1406.5155].

[123] F. Cachazo, S. He and E. Y. Yuan, *New Double Soft Emission Theorems*, *Phys. Rev.* **D92** (2015) 065030 [1503.04816].

[124] P. Di Vecchia, R. Marotta and M. Mojaza, *Double-soft behavior for scalars and gluons from string theory*, *JHEP* **12** (2015) 150 [1507.00938].

[125] K. A. Roehrig and D. Skinner, *A Gluing Operator for the Ambitwistor String*, *JHEP* **01** (2018) 069 [1709.03262].

[126] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014.

[127] N. Arkani-Hamed, T.-C. Huang and Y.-t. Huang, *Scattering Amplitudes For All Masses and Spins*, 1709.04891.

[128] P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, *Multiple Bremsstrahlung in Gauge Theories at High-Energies. 1. General Formalism for Quantum Electrodynamics*, *Nucl. Phys.* **B206** (1982) 53.

[129] F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, *Multiple Bremsstrahlung in Gauge Theories at High-Energies. 2. Single Bremsstrahlung*, *Nucl. Phys.* **B206** (1982) 61.

[130] R. Kleiss and W. J. Stirling, *Spinor Techniques for Calculating* $p\bar{p} \to W^{\pm}/Z^0$ *+ Jets*, *Nucl. Phys.* **B262** (1985) 235.

[131] Z. Xu, D.-H. Zhang and L. Chang, *Helicity Amplitudes for Multiple Bremsstrahlung in Massless Nonabelian Gauge Theories*, *Nucl. Phys.* **B291** (1987) 392.

[132] R. Gastmans and T. T. Wu, *The Ubiquitous photon: Helicity method for QED and QCD*, *Int. Ser. Monogr. Phys.* **80** (1990) 1.

[133] A. Marzolla, *The 4D on-shell 3-point amplitude in spinor-helicity formalism and BCFW recursion relations*, *PoS* **Modave2016** (2017) 002 [1705.09678].